This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

SOLUTIONS OF EXAMPLES

IN

CONIC SECTIONS,

TREATED GEOMETRICALLY

BY

W. H. BESANT, M.A., F.R.S.

LECTURER OF ST JOHN'S COLLEGE.

BIBLIOTHECA
AUG 1882
BODLEIANA

CAMBRIDGE:
DEIGHTON, BELL AND CO.
LONDON: GEORGE BELL AND SONS.
1881

183 . 9 . 82

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

PREFACE TO SOLUTIONS.

I HAVE frequently received requests for a book of Solutions of the Examples in my treatise on Conic Sections, but have never been able to find time to prepare them.

Mr Archer Green, B.A., Scholar of Christ's College, volunteered to undertake the task, with the aid of my notes and his own, and, with the exception of a few at the end, wrote out the solutions entirely.

Mr Green was however prevented by illness from completing the revision of the proofs, and I am much indebted to Mr J. Greaves, Fellow of Christ's College, who kindly undertook to examine the rest of the sheets.

The book will, I hope, prove useful both to students and teachers, as a companion volume to the treatise on Conic Sections.

W. H. BESANT.

Sept. 1881.

CONIC SECTIONS.

SOLUTIONS OF EXAMPLES.

CHAPTER I.

1. IF the tangent at P meet the directrix in Z, and S be the focus, PSZ is a right angle;

$\therefore S$ lies on the circle of which PZ is diameter.

2. Let PN and QM be the ordinates at P and Q.

Then $PN : QM :: SP : SQ :: XN : XM$;

$\therefore$ the triangles PXN and QXM are similar and PX, QX equally inclined to XS.

3. By Art. 8, FS is the external bisector of the angle PSQ.

4. $SP : PK :: SA : AX :: SE : EK$;

$\therefore EP$ bisects the angle SPK.

5. Since F, S, P and K lie on a circle,

the angle KSF = the angle FPK = the angle FTS.

6. $PN : P'N' :: SP : SP'$;

$\therefore XK : XN :: XK' : XN'$;

$\therefore$ the angle LNN' = the angle $K'N'X$ = the angle $LN'N$.

7. Let Q be the point where the tangent at R meets NP.

Then $NQ : NX :: SR : SX :: SA : AX :: SP : NX$;

$$\therefore SP = QN.$$

8. Let SY be perpendicular to the tangent at P and GL perpendicular to SP.

Then, since the triangles PSY, GPL are similar,

$$PG : PL :: SP : SY,$$

or

$$PG : SR :: SP : SY.$$

9. If the tangent meet the directrix in Z, and SP be drawn such that ZSP is a right angle meeting the tangent in P,

then P will be the point of contact of the tangent ZP.

10. If P, Q be the extremities of the chord, and PK, QL be perpendicular to the directrix,

$$SP : PK :: SA : AX :: SQ : QL;$$

$$\therefore SP + SQ :: PK + QL :: SA : AX.$$

Now the distance of the middle point of PQ from the directrix is equal to half $PK + QL$, and is therefore least when $SP + SQ$ is least, that is, when PQ goes through the focus.

11. If TP, TP' be the fixed tangents, and the tangent at Q meet them in E, E',

the angle PSE = the angle ESQ, and the angle QSE' = the angle $E'SP'$;

$\therefore$ the angle ESE' = half the angle PSP'.

12. If perpendiculars from the given points PK, QL be drawn to the directrix and S be the focus,

$SP : SQ :: PK : QL$, a constant ratio;

$\therefore$ the locus of S is a circle.

13. If O be the fixed point on the axis,

$$SO : SR :: SG : SP :: SA : AX;$$

$\therefore SR$ is constant and R lies on a circle of which S is centre.

14. $$AT : AX :: SR : SX :: SA : AX;$$
$$\therefore AT = AS.$$

15. If EA meet the curve in P, then, see fig. Art. 1, KS is perpendicular to ES;

$$\therefore E' \text{ and } K \text{ coincide};$$

$$\therefore \text{the angle } KSP = KSX = XES = FSE;$$

$\therefore FSP$ is a right angle, since ESE' is a right angle;

$$\therefore FP \text{ is a tangent.}$$

Similarly $E'A$ meets the curve in the point of contact of the other tangent which passes through F.

16. $$AT : SR :: EA : ER :: AX : SX;$$
$$\therefore AT : AX :: SR : SX :: SA : AX;$$
$$\therefore AT = AS.$$

17. If TL be perpendicular to the directrix,

$$SR : TL :: SA : AX :: SM : TL;$$
$$\therefore SM = SR.$$

18. FS is the external bisector of the angle QSP, and $F'S$ of QSP';

$$\therefore \text{the angle } FSF' = \text{half the angle } PSP'.$$

19. Since the triangles SPN, SGL are similar,

$$\therefore GL : PN :: SG : SP :: SA : AX.$$

20. If the normals PG, $P'G'$ meet in Q, and QV be drawn parallel to the axis to meet the chord in V,

$$VQ : VP :: SG : SP :: SA : AX :: SG' : SP' :: VQ : VP';$$
$$\therefore VP = VP', \text{ or } V \text{ bisects } PP'.$$

21. DS is the external bisector of the angle PSQ, and ES of pSQ;

$\therefore DSE$ is a right angle.

22. The latus rectum is a harmonic mean between SP and SP';

$$\therefore 2SP \cdot SP' = SR \cdot PP'.$$

23. $PE : PL :: PQ : PG :: PV : PS :: PP' : 2SP$, see Ex. 20;

$$\therefore PE : SR :: SP' : SR;$$

$$\therefore PE = SP'.$$

Similarly $P'E = SP$.

24. The right-angled triangles DSQ, DSE have a common hypotenuse.

Also $SE = SR = SQ$;

$\therefore$ the angle QSE = the angle ESP.

25. Let S be the focus and P and Q the given points.

Through P draw a straight line PK so that SP may bear to PK the given ratio of the eccentricity.

Through Q draw a straight line QL so that $SQ : QL$ in the same ratio.

With centres P, Q and radii PK, QL respectively describe circles.

The perpendicular from S on a common tangent to these circles will be axis.

26. Let the tangents at P and Q intersect in T.

Draw TN perpendicular to directrix and TM perpendicular to SP.

Then $SM : TN :: SA : AX$.

But ST bears a constant ratio to SM, since angle TSM = half PSQ;

$\therefore ST$ bears a constant ratio to TN.

27. Let T be the intersection of the tangents at P and p.

Draw TK perpendicular to Pp.

Then $TK : PL :: TP : PG$ and $TK : pl :: Tp : pg$.

Again, draw GM, gm perpendicular to SP, Sp respectively, and TN, Tn perpendicular to SP, Sp respectively.

Then $TP : PG :: TN : MP :: Tn : mp :: Tp : pg$;

$\therefore TK : PL :: TK : pl$;

$\therefore PL = pl$.

CHAPTER II.

THE PARABOLA.

1. THE distance of the centre of the circle from the fixed point is equal to its distance from the fixed straight line, and therefore its locus is a parabola of which the fixed point is focus and the fixed straight line directrix.

2. Through the vertex draw a straight line making the given angle with the axis; the tangent at the point where the diameter bisecting this chord meets the curve will be the tangent required.

Or, draw a radius vector from the focus, making twice the given angle with the axis.

3. Since $TA=AN$, $PN=2AY$; $\therefore AY^2=AS.AN$.

4. Let SY' be drawn perpendicular to the line through G parallel to the tangent.

Then in the right-angled triangles YST, $Y'SG$, $ST=SG$, and the angles YST, $Y'SG$ are equal;

$$\therefore SY=SY'.$$

5. Draw SY perpendicular to the tangent and YA perpendicular to the axis.

Produce SA to X, making AX equal to SA.

Then the straight line through X perpendicular to SX is the directrix.

6. Let the circle touch the fixed circle in Q, and the straight line in R; let P be its centre, and S the centre of the fixed circle.

Produce PR to M, making RM equal to SQ, then the

straight line MX drawn through M parallel to the given line is a fixed straight line.

Then, since SP is equal to PM, the locus of P is a parabola of which S is focus and MX directrix.

7. Draw SY, SY' perpendiculars on the two tangents.

Then, if SA be perpendicular to YY', A is the vertex.

Produce SA to X, making AX equal to AS; X is the foot of the directrix.

8. If the tangent at the end of the latus rectum meet PN in Q,

$$QN = XN = SP.$$

9. Since SYP and PNS are right angles, P, N, S, Y lie on a circle;

$$\therefore TY \cdot TP = TS \cdot TN.$$

10. SE is half TP,

and $$PT^2 = PN^2 + TN^2 = 4AS \cdot AN + 4AN^2;$$

$$\therefore SE^2 = AN \cdot XN = AN \cdot SP.$$

11. If SY be drawn perpendicular to the tangent and A be vertex, SAY is a right angle;

$\therefore A$ lies on the circle of which SY is diameter.

12. Draw SY perpendicular to the tangent, then if the circle described with centre S and radius equal to a quarter of the latus rectum meet the circle described on SY as diameter in A, A is the vertex.

Produce SA to X, making AX equal to SA, then X is the foot of the directrix.

13. $$SN : SP :: SN' : SP',$$

or $$AN - AS : AN + AS :: AS - AN' : AS + AN';$$

$$\therefore AN : AS :: AS : AN';$$

$$\therefore AN \cdot AN' = AS^2.$$

Again, $4AS.AN : 4AS^2 :: 4AS^2 : 4AS.AN'$;

$$\therefore PN^2 : SR^2 :: SR^2 : P'N'^2,$$

or $$PN : SR :: SR : P'N';$$

$\therefore$ the latus rectum is a mean proportional between the double ordinates.

14. Let V be the middle point of the focal chord PSP', and let the diameter through V meet the curve in Q; then, if QT, QM be the tangent and ordinate at Q, and VL be ordinate of V,

$$VL = QM \text{ and } TM = SL;$$

$$\therefore VL^2 = QM^2 = 4AS.AM = 2AS.TM = 2AS.SL.$$

Hence the locus of V is a parabola of which S is vertex and SL axis.

15. If P, P' be the given points, PK, $P'K'$ perpendiculars on the directrix, the focus is the point of intersection of a circle centre P, radius PK with a circle of which P' is centre and $P'K'$ radius.

In general two circles intersect in two points, therefore two parabolas can be drawn satisfying the given conditions.

16. If PG be normal at P, the triangles PNG, pPR are similar;

$$\therefore Pp : PN :: RP : NG;$$

$$\therefore RP = 2NG = \text{latus rectum};$$

$\therefore$ the locus of R is an equal parabola having its vertex A' on the opposite side of X, such that AA' is equal to the latus rectum.

17. Let P, P' be the given points, S the given focus.

A common tangent to the circles described with centres P, P' and radii PS, $P'S$ respectively will be the directrix.

18. If SP be the focal distance and SY perpendicular to the tangent at P, Y lies on the circle of which SP is diameter.

Also the angle AYS = the angle SPY;

$\therefore$ AY touches the circle.

19. The tangents at the ends of the focal chord PSP' meet in F on the directrix at right angles: also the straight line through F at right angles to the directrix bisects PP' in V;

$$\therefore FV = VP = VP';$$

$\therefore$ the directrix touches the circle described on PP' as diameter.

20. Draw the farther tangent to the circle parallel to the given diameter, then the locus of the point is a parabola of which the centre is focus, and the tangent thus drawn directrix.

21. Draw a straight line parallel to the given straight line, on the farther side of it, and at a distance from it equal to the radius of the circle, then the locus of the point is a parabola of which the centre of the circle is focus, and the straight line thus drawn directrix.

22. Let Q be the centre of the circle touching the sector in R and AC in M.

Through C draw CB at right angles to AC, and on the same side of it as Q, and draw QN perpendicular to the tangent at B.

Then $$NQ + QM = BC = CQ + QR;$$

$$\therefore CQ = QN;$$

$\therefore$ Q lies on a parabola of which C is focus and BN directrix.

23. Y is the middle point of TP and Z of PG;
therefore YZ is parallel to the axis.

24. If SQ be perpendicular to the normal PG,

$$PQ = QG,$$

and if QM be the ordinate, $NM = MG$;

$$\therefore SM = AN \text{ and } PN = 2QM;$$

$$\therefore QM^2 = AS.AN = AS.SM;$$

$\therefore$ Q lies on a parabola of which S is vertex and SG axis.

25. The triangle PSG is isosceles; therefore GL is equal to PN.

26. If the circle described with centre S and radius equal to the perpendicular from S on the tangent at P meet the circle of which SP is diameter in Y, and the angle SYA be made equal to the angle SPY, then the foot of the perpendicular SA on YA will be the vertex.

27. Since SQ is double SA, ASQ (and likewise QSP) is equal to the angle of an equilateral triangle; therefore SP and SQ are equally inclined to the latus rectum.

28. $$QX^2 = SX^2 + SQ^2 + 2SX.SQ$$
$$= 4AS^2 + QG^2 + 2SQ.NG$$
$$= 4AS^2 + QN^2 + 2QN.NG + NG^2 + 2SQ.NG$$
$$= 4AS^2 + QN^2 + NG^2 + 2NG.SN$$
$$= 4AS^2 + NQ^2 + 2AN.NG$$
$$= 4AS^2 + QN^2 + PN^2 = 4AS^2 + QP^2.$$

29. The angle
$$SPF = SPG - FPG = SGP - GPH = SHP;$$
therefore the triangles SPF, SHP are similar;
$$\therefore SF.SH = SP^2 = SG^2.$$

30. A, B, C, S lie on a circle; therefore, if D be the end of the diameter drawn through S, DA, DB, DC are perpendicular to SA, SB, SC respectively.

31. Since PQ and PR are equally inclined to the axis, the circle through P, Q, R touches the parabola at P; therefore PQ is a diameter of this circle.

Therefore PRQ, the angle in a semicircle, is a right angle.

32. Let MR and AQ meet in V.

Draw the ordinates VW, RZ.

Then $MW : MZ :: WV : RZ :: AW : AN;$

$\therefore MW : AW :: MZ : AN;$

$\therefore AN : AW :: AZ : AN.$

Again, $VW^2 : QN^2 :: AW^2 : AN^2;$

$\therefore VW^2 : RZ^2 :: AW : AZ;$

$\therefore V$ lies on the curve.

33. Let P, Q be the given points. Bisect PQ in V, and draw VT parallel to the axis meeting the given tangent P in T.

Draw PS, QS such that TP, TQ may be equally inclined to the axis and to SP, SQ respectively. PS, QS meet in the focus.

Through P draw a straight line PK parallel to the axis, making PK equal to SP, then the straight line through K at right angles to PK will be the directrix.

34. Let P be the vertex and QVQ' be corresponding ordinate.

Take M in VP produced such that

$$QV^2 = 4MP \,.\, PV.$$

Make angle TPS equal to the angle MPT, PT being parallel to QQ', and make PS equal to PM.

Then S is the focus, and the straight line drawn through M at right angles to PM is the directrix.

35. $PM^2 : QN^2 :: AM : AN$, QN being the ordinate of Q;

$\therefore AM = 4AN$ and $3AM = 4NM$;

$\therefore 3AT = 4QN = 2PM.$

36. Draw PN perpendicular to AB.

Then $AN : NP :: CQ : AC :: NP : AC;$

$\therefore PN^2 = AC \,.\, AN.$

Therefore the locus of P is a parabola of which A is vertex and AB axis.

37. The triangles LKP, PSK, KSA and TKA are similar;

$$\therefore KL^2 : SP^2 :: KP^2 : KS^2 :: KA^2 : AS^2 :: TA : AS$$
$$:: SP - AS : AS.$$

38. With centre S and radius one-fourth of the chord describe a circle meeting the parabola in P. The chord through S parallel to the tangent at P will be the chord required.

39. $$PN^2 = 4AS.AN = 4AS.AN' + 4AS^2$$
$$= P'N'^2 + N'G'^2 = P'G'^2.$$

40. If Pp, $P'p'$ be two parallel chords, and V, V' their middle points, VV' is a diameter. Let VV' meet the curve in Q.

Draw QT parallel to Pp, then QT is the tangent at Q.

Produce VV' to a point M such that $PV^2 = 4QM.QV$, then the straight line drawn through M at right angles to MV is the directrix.

Make the angle TQS equal to the angle TQM.

Then if QS be made equal to QM, S is the focus and the straight line through S perpendicular to the directrix is the axis.

41. Let the tangents at P and P' intersect in T.

Then $$4SP.PV : 4SP'.P'V'$$
$$:: P'V^2 : PV'^2 :: TP^2 : TP'^2 :: SP : SP';$$
$$\therefore PV = P'V'.$$

42. If in the preceding Example $P'T$ meets PV in Z and the sides of the triangle ABC are parallel to ZP, PT and TZ respectively,

$$AB : AC :: TZ : TP :: TP' : TP.$$

43. If U, V be the vertices of the diameters bisecting Pp, Qq,

$$PS.Sp : QS.Sq :: SU : SV :: Pp : Qq.$$

44. Draw RW, LZ parallel to QQ'.

Then $$PL^2 : PR^2 :: LZ^2 : RW^2$$
$$:: QV^2 : RW^2 :: PV : PW :: PN : PR;$$
$$\therefore PL^2 = PR \cdot PN.$$

45. This question is solved in Conics, Art. 212, p. 217.

46. $$PN^2 : AN^2 :: AM^2 : QM^2;$$
$$\therefore 4AS : AN :: AM : 4AS.$$

47. Let AP, Ap meet the latus rectum in L and l respectively.

Then $PN^2 : SL^2 :: AN^2 : AS^2 :: AN : An$, (Example 13)
$$:: PN^2 : pn^2;$$
$$\therefore SL = pn.$$
In like manner $$Sl = PN.$$

48. If PK, QL be perpendicular to the directrix, and QL' to PK produced, the angle SPQ = the angle QPL';
$$\therefore PL' = SP = PK;$$
$$\therefore SQ = QL = KL' = 2PK = 2SP.$$

49. Is equivalent to Example 32.

50. Let Q be the point of intersection, and let QK be the ordinate of Q.

Then $$AK : QK :: PN : NT;$$
$$\therefore 2AK \cdot AN = QK \cdot PN = PN^2 = 4AS \cdot AN;$$
$$\therefore AK = 2AS,$$
or Q lies on a fixed straight line parallel to the directrix.

51. Let Tp, Tq be the fixed tangents, and let PQ touch the curve in R.

Then $$SP^2 = Sp \cdot SR = Sq \cdot SR = SQ^2;$$
$$\therefore SP = SQ.$$

52. Let TM be the ordinate of T, and TW perpendicular to SP.

$$TM^2 = ST^2 - SM^2 = TW^2 + SW^2 - SM^2$$
$$= TW^2 + XM^2 - SM^2 = TW^2 + XS^2 + 2XS.\ SM;$$

$\therefore$ the locus of T is a parabola of which XS is axis.

If $TW = 2AS$, $TM^2 = 4AS.\ XM$, or X is the vertex.

53. Let the chord PQ meet the axis in O, and the tangent at A in V.

Then by Art. 48, $VO^2 = VP.\ VQ$;

$\therefore$ V is a fixed point, and the locus of A is the circle of which OV is diameter.

54. Let the diameter TV meet the curve in R.

Then the tangent at R, being parallel to PQ, meets TP at right angles in Z on the directrix.

Also $$TZ : ZP :: TR : RV;$$
$$\therefore\ TZ = ZP.$$

Therefore T and P are equidistant from the directrix.

55. Let PT meet the axis in t.

Then $$PQ : PT :: 2PV : PT :: 2PG : Pt,$$
$$:: 2PN : Nt :: PN : AN.$$

56. If the tangents TP, TQ are equal, T lies on the axis.

Let the tangent at R meet them in p and q.

Then, since T, p, q and S lie on a circle, the triangles SqT, SpP are similar;

$$\therefore\ Tq : pP :: TS : SP;$$
$$\therefore\ Tq = pP.$$

So $$Tp = qQ.$$

57. $AN.NL=PN^2=4AS.AN;$

$$\therefore NL=4AS.$$

But $NG=2AS;$

$\therefore LG$ = half the latus rectum.

58. By Art. 5, $P'S$, $Q'S$ are the external bisectors of the angles PSA, QSA;

therefore $P'SQ'$ is a right angle.

59. The angles TCS, DRS are equal, being supplements of equal angles SCP, SRC, Art. 35.

And the angle $CTS=TQS=RDS;$

$\therefore$ the triangles TCS, DRS are similar;

$$\therefore DR : TC :: RS : SC :: RC : CP;$$

$$\therefore PC : CT :: CR : RD.$$

Similarly $TD : DQ :: CR : RD.$

60. Let AD and XP intersect in Q, and let QM be the ordinate.

Then $QM : DS :: AM : AS$ and $QM : PN :: XM : XN;$

$$\therefore AM : XM :: AS : XN;$$

$$\therefore AM : AS :: AS : AN;$$

$$\therefore QM^2 : PN^2 :: AM^2 : AS^2 :: AM : AN,$$

or Q is on the parabola.

61. By Example 18, YY', the tangent at the vertex, is a common tangent.

$$SY^2=AS.SP, \qquad SY'^2=AS.Sp;$$

$$\therefore YY'^2=SY^2+SY'^2=AS.Pp.$$

62. If PV be the diameter bisecting AQ,

$$AM=4AN.$$

Also $$AM.MR = QM^2 = 4AS.AM;$$
$$\therefore\ MR = 4AS.$$

Now focal chord parallel to AQ
$$= 4SP = 4XN = 4AS + AM = AR.$$

63. Let AR, CP meet in p.

Draw pN, pD perpendicular to CA, CR, and let Dp meet the tangent at A in M.

$$Cp : CP :: CD : CR :: Np : CR :: AN : AC;$$
$$\therefore\ Cp = AN = pM.$$

Therefore the locus of p is a parabola of which C is focus and AM directrix.

64. If QMQ' be the common chord,

$$9AS^2 = 4AQ^2 = 4AM^2 + 4QM^2 = 4AM^2 + 16AM.AS;$$
$$\therefore\ AM \text{ is half } AS.$$

65. Let the fixed straight line BK meet the tangent at P in K.

Draw KY' at right angles, and SY' parallel to KP.

Draw $Y'A'$ perpendicular to the axis, and KL parallel to BA.

Then, since $KY = SY'$, $SA' = KL = BA$

therefore A' is a fixed point.

Therefore KY' touches the parabola of which S is focus and A' vertex.

66. $$QD.DR = QM^2 - DM^2 = QM^2 - PN^2$$
$$= 4AS.AM - 4AS.AN = 4AS.PD.$$

67. Draw the double ordinate QMq; then, if the diameter through Q' meet Qq in D',

$$QD'.D'q = 4AS.Q'D'.$$

Now $NT : PN :: QD' : QD' :: D'q : 4AS$;

$\therefore D'q : 4AS :: 2AN : PN :: PN : 2AS$;

$\therefore 2PN = D'q = D'M + Mq = QM + Q'M'$.

Therefore the line through P bisecting QQ' is parallel to the axis.

Hence the locus of the middle points of a series of parallel chords is a straight line parallel to the axis.

68[1]. Take CP, CQ two tangents such that PCQ is two-thirds of a right angle; join SC cutting the curve in R, and draw the tangent ARB. Then, Art. 38,

$CSP = CSQ = 120^0$, and $CAR = \frac{1}{2}CSQ = 60^0$;

$\therefore CAB$ is equilateral.

69. Draw AZ, AN perpendicular to the tangent and SY respectively, and draw SM perpendicular to ZA.

Then $SM^2 = AN^2 = YN . NS = ZA . AM$.

Therefore the locus of S is a parabola of which A is vertex and ZM axis.

70. If GZ be drawn parallel to PY and SZ to PG, then SY, SZ are equal.

Therefore, if ZB be perpendicular to the axis, $BS = AS$.

Hence GZ touches an equal parabola of which B is vertex and S focus.

71. If pqr be the triangle formed by the given straight lines, describe a parabola passing through p, q and r having its axis parallel to AS. (Ex. 45.)

If s be the focus of this parabola, draw SP parallel to sp, PQ to pq, and PR to pr.

Then $PQ : pq :: SA : sa :: PR : pr$,

and the angles QPR, qpr are equal.

[1] If a parabola touch the sides of an equilateral triangle, the focal distance of any vertex of the triangle passes through the point of contact of the opposite side.

72. Let RW be the ordinate of R.

Then

$$AN^2 : AW^2 :: PN^2 : RW^2 :: PN^2 : QM^2 :: AN : AM;$$

$$\therefore AN : AW :: AW : AM,$$

or

$$WN : AN :: MW : AW;$$

$$\therefore RL : QR :: AN : AW :: PN : NL.$$

73. Let PV be the ordinate to the diameter RQM.

Then

$$PM : RM :: PN : TN$$

$$:: 2PN . AS : 4AS . AN :: 2AS : PN;$$

$$\therefore PM . PN = 2AS . RM.$$

But

$$PM^2 = 4AS . QV = 4AS . RQ;$$

$$\therefore RM : RQ :: 2PN : PM :: PP' : PM;$$

$$\therefore QM : QR :: P'M : PM.$$

74. Let PP' be the chord, TWV its diameter, RQM the line parallel to the axis.

Then

$$PM : RM :: PV : TV :: PV : 2WV$$

$$:: 2VP . SW : 4SW . WV :: 2SW : PV;$$

$$\therefore PM . PV = 2SW . RM.$$

But

$$PM . MP' = 4SW . QM,$$

$$\therefore RM : QM :: 2PV : MP',$$

or

$$RQ : QM :: PM : MP'.$$

75. SR, Sr are the exterior bisectors of the angles PSQ, pSQ respectively.

Therefore RSr is a right angle.

Therefore SD, which is half the latus rectum, is a mean proportional between DR and Dr.

76. Let PVP' be parallel to the given straight line, $QV'Q'$ the chord joining the two other points of intersection of the parabola and circle.

Let the diameters through V and V' meet the curve in p and p'.

Then pp' is a double ordinate; draw $V'H$ parallel to pp' to meet pV.

VV' is perpendicular to QQ', and therefore parallel to the normal at p';

$$\therefore\ VV' : p'g :: V'H : p'n;$$
$$\therefore\ VV' = 2p'g.$$

77. The arcs QU and RV are equal, since QV and UR are parallel.

Therefore QR and UV are equally inclined to QV, that is to the axis.

But QR and the tangent at P are equally inclined to the axis;

therefore UV is parallel to the tangent at P.

78. $$VR : VR' :: VR : V'Q' :: PV : PV'$$
$$:: QV^2 : Q'V'^2 :: QV^2 : VR'^2;$$
$$\therefore\ VR \cdot VR' = QV^2.$$

79. If FR, QE meet the tangent at P in V and T,

$$TE : EQ :: VR : RF :: PF : FQ. \quad \text{(Ex. 74.)}$$

Therefore EF is parallel to TP.

80. If Q be the vertex of the diameter bisecting the chord Rr which meets the diameter PW in W,

$$RW \cdot Wr = 4SQ \cdot PW.$$

Therefore the rectangle under the segments varies as the distance of the point of intersection W from P.

81. QS, $Q'S$ are equally inclined to SP, and therefore to the axis.

Therefore $Q'S$ meets the curve at the end of the double ordinate QMq, and, since $AM \cdot AM = AS^2$, the semi-latus rectum is a mean proportional between QM and $Q'M'$.

Also, since the diameter through P bisects QQ', PS is an arithmetic mean between QM and $Q'M'$.

82. BB' will bisect CA' in V.

Let V' be the middle point of $B'B''$.

VV' is parallel to the axis.

And BB'' is parallel to VV', and therefore to the axis.

Similarly AA'' and CC'' are parallel to the axis.

83. Let C be the centre of the circle.

The angle between tangents to circle $= PCP' = 2PSP'$ $= 4$ times angle between the tangents to the parabola.

84. The tangents at the ends of the focal chord PSP' will meet in T on the directrix.

If the normals at P and P' meet in Q, TQ will be parallel to the axis.

Let TQ meet the curve in p and PP' in V. Let QM be the ordinate of Q.

Then $$XM = TQ = 2TV = 4Sp = 4Xn.$$

Therefore, if we take B in XM such that $XB = 4AS$,

$$BM = 4An, \quad QM^2 = pn^2 = 4AS \cdot An = AS \cdot BM.$$

Hence the locus of Q is a parabola of which B is vertex and BM axis.

85. Produce PA to P', making AP' equal to AP.

On AP' as diameter describe a circle meeting the tangent at P in T.

Join TA and produce to N, making AN equal to AT.

In AN take a point S such that $PN^2 = 4AS \cdot AN$, then S is focus.

86. If G be the intersection of the normals and Q vertex of the diameter bisecting PSp,

$$PS \cdot Sp = AS \cdot Pp = AS \cdot TG.$$

87. If pq be a tangent parallel to PQ, $Tp = pP$, and T, p, q, S and O lie on a circle.

Therefore the angles TSO, TpO are equal, and TpO is a right angle.

88. $SM^2 : AN^2 :: QM^2 : PN^2 :: AM : AN;$

$$\therefore SM^2 = AM.AN.$$

So $$SM'^2 = AM'.AN;$$

$$\therefore MM'.AN = MM'.(SM - SM');$$

$$\therefore SM - SM' = AN.$$

$$MM' = SQ - SQ';$$

$$\therefore MM' : SM - SM' :: AP : AN;$$

$$\therefore MM' = AP.$$

89. If P, Q, P', Q' be the points of intersection, PQ, $P'Q'$ are equally inclined to the axis.

Hence the middle points of PQ and $P'Q'$ are equidistant from the axis.

Therefore, if P, Q be on one side of the axis and $P'Q'$ on the other, the sum of the ordinates of P and Q is equal to the sum of the ordinates of P' and Q'.

If P' be on the same side of the axis as P and Q, the ordinate of Q' is equal to the sum of the ordinates of P, Q, and P'.

90. Let the diameter through T meet the curve in W, PQ in V, and PN in t.

Let WZ be the ordinate of W; draw Qq parallel to the axis to meet PN.

$$QM.PN = PN.qN = tN^2 - Pt^2 = WZ^2 - 4AS.WV$$
$$= 4AS.AZ - 4AS.LZ = 4AS.AL.$$

91. $pX : XA :: PN : AN :: 4AS : PN$

$$:: 4AS.QM : 4AS.AL. \quad \text{(Ex. 90.)}$$

So $$qX : XA :: PN : AL;$$

$$\therefore pX + qX : XA :: PN + QM : AL;$$

$$\therefore pX + qX : PN + QM :: XA : AL :: tX : TL.$$

But $$NP + QM = 2TL;$$

$$\therefore pX + qX = 2tX,$$

or $$pt = tq.$$

92. Let TF, TD be drawn parallel to PE, QE normals at P and Q.

The angle $TFQ = PEQ =$ supplement of $PTQ = TSQ$;

$\therefore$ Q, S, F, T lie on a circle.

Therefore TSF is a right angle.

So TSD is a right angle, and DF goes through S.

93. If pq be a tangent parallel to PQ, $Tq = qQ$.

Also, T, p, q and S lie on a circle;

therefore the angles Tpq, TSq are equal.

Therefore TSq is a right angle.

94. Let RO be the diameter through the given point O.

Take T in OR produced such that $TR : RO$ in the given ratio.

If TP be a tangent, the chord POQ will be divided as required. (Ex. 74.)

95. If QN be the ordinate,

$$BP + PQ = QN + BX - NX = BX + QN - SQ,$$

which is greatest when $QN = SQ$, that is when Q is on the latus rectum.

96. If SZ and PG meet in Q and QT be ordinate,

$$TA : AS :: QZ : ZS :: QP : PG :: TN : NG;$$

$$\therefore\ TN = 2TA.$$

97. If QV be the ordinate of the point of contact,

$$TP = PV.$$

Therefore the distance of V from TQ is twice the distance of P, or the locus of V is a straight line parallel to TQ.

98. If $TPSQ$ be the parallelogram, the angles TSP, TSQ are equal;

therefore $TPSQ$ is a rhombus and T lies on the axis.

Therefore TSP is the angle of an equilateral triangle.

99. If SZ, SZ' be the perpendiculars on the second tangents, TQ, TQ' and PP' be the common tangent, SY perpendicular to it,

then angle $A'SY = ASY = YSP$;

$\therefore$ A' lies in SP, and A in SP';

$\therefore$ $SP = SP'$.

Now $SQ \cdot SP = ST^2 = SQ' \cdot SP'$;

$\therefore$ $SQ = SQ'$;

$\therefore$ $SZ = SZ'$.

100. If the tangent meet AY in Y and the other parabola in Q,

$$QM^2 = \tfrac{1}{2} AS \cdot AM, \quad AY^2 = AS \cdot AT,$$

$$QM : AY = MT : AT;$$

$$\therefore 2TM^2 = AT \cdot AM.$$

This can be constructed by taking $AM = MT$, or by taking $AM = 2AT$, the two solutions corresponding to the two points in which the parabola is cut by the tangent.

CHAPTER III.

THE ELLIPSE.

1. $SD^2 = BC^2 = CS \cdot SX$.

Therefore CDX is a right angle.

2. ST, SP are equally inclined to PT, since pST is parallel to $S'P$.

Therefore $ST = SP$.

3. $PN : PG' :: SY : SP :: BC : CD$
$:: PF : AC :: AC : PG'$.

Therefore $PN = AC$.

4. T lies on a circle of which QQ' is a diameter and V centre;

therefore $VT = VQ$.

Now $QV^2 : CP^2 - CV^2 :: CD^2 : CP^2$,

or $VT^2 : CV \cdot CT - CV^2 :: CD^2 : CP^2$.

Therefore $TV : VC :: CD^2 : CP^2$.

Therefore $CT : CV :: CD^2 + CP^2 : CP^2$,

or $CT^2 : CV \cdot CT :: AC^2 + BC^2 : CP^2$.

Therefore $CT^2 = AC^2 + BC^2$.

5. Through T draw a straight line at right angles to AA' meeting AP, $A'P$ in E, E'.

Then $ET : PN :: AT : AN :: CT - CA : CA - CN$.

Now $$CT : CA :: CA : AN;$$
$$\therefore CT+CA : CT-CA :: CA+AN : CA-CN;$$
$$\therefore ET : PN :: A'T : A'N :: ET : PN.$$

Hence PT bisects any straight line parallel to ET terminated by $A'P$, AP.

6. Draw CD parallel to the given line, and CP parallel to the tangent at D.

The tangent at P will be parallel to CD and the given line.

7. $$SR : XE :: SA : AX :: SR : SX.$$
Therefore $$XE=SX,$$
and $$AT=AS.$$

8. Draw GL perpendicular to SP.

Then $$PL=SR,$$
and $$SY : SP :: PL : PG :: SR : PG.$$

The angle SPS' is greatest when SPY is least, that is when $SY : SP$ or $BC : CD$ is least.

Hence SPS' is greatest when CD is greatest, that is when
$$CD=AC.$$
Hence SPS' is greatest when P is on the minor axis.

9. $$CE'^2=CP^2+PE'^2+2PF.PE'=CD^2+CP^2$$
$$+2CD.PF=AC^2+BC^2+2AC.BC;$$
$$\therefore CE'=AC+BC.$$
So $$CE=AC-BC.$$
$$(CP+CD)^2=AC^2+CB^2+2CP.CD,$$
which is greater than $(AC+BC)^2$, since $CP.CD$ is greater than $PF.CD$ or $AC.BC$.

Similarly $CP\sim CD$ is less than $AC-BC$.

10. Let $S'Q$ drawn parallel to SP meet the normal in K, and SY in Q.

Then $S'K = S'P$ and $KQ = SP$;

therefore $S'Q = AA'$.

11. $SY : S'Y' :: YP : PY' :: TP - TY : TY' - TP$
$:: PG - SY : S'Y' - PG.$

12. $PS'Q$ is the supplement of $QPS' + PQS'$,

and is therefore equal to the excess of twice $QPT + PQT$ over two right angles,

that is, is the supplement of twice PTQ.

13. Since CZ and SP are parallel, the angle
$CZP = SPY = SNY$;

therefore Y, Z, C, N lie on a circle.

14. Let AQ and SP meet in R.

Then $SA : SR :: SG : SP :: SA : AX$.

Therefore R lies on a circle of which S is centre.

15. Since KPt is a right angle, t lies on a circle which passes through S, P, S', K,

therefore $GK : SK :: S'G : S'P :: SA : AX$,

and $St : tK :: SY : SP :: BC : CD$.

16. If SP meet $S'Y'$ in Z, then since $S'Y' = Y'Z$, SY' will bisect PG.

17. Let the circle whose centre is P touch the circles whose centres are S, H in Q, R.

Then $SP + PH = SQ + QP + PH = SQ + HR$.

Hence the locus of P is an ellipse of which S and H are foci.

18. $TN : TC :: PN : Ct$.

Therefore $TN.NG : CT.NG :: PN^2 : Ct.PN$.

But $PN^2 = TN.NG$.

Therefore $CT.NG = Ct.PN = CB^2$.

19. $TP : TQ :: CD : AC :: BC : PF :: PG : BC.$

20. $PN^2 : AF.A'F :: TN^2 : TA \,.\, TA'$
$:: TN^2 : CT^2 - CA^2 :: TN : CT$
$:: CT - CN : CT :: CA^2 - CN^2 : CA^2;$
$\therefore AF.A'F' = BC^2.$

21. The perpendiculars from T on SP, SQ, HP, HQ are all equal.

Hence a circle can be described with centre T to touch SP, SQ, HP, HQ.

22. If P, Q be two points of intersection, PC bisects the angle ACa and QC bisects $A'Ca$.

Therefore PCQ is a right angle.

23. If SP, HQ meet in R,

$PSQ + PHQ = 2PRQ - SQH - SPH,$

and $SQH + SPH + 2RQT + 2RPT = 4$ right angles,

$\therefore PSQ + PHQ =$ twice the supplement of QTP.

24. Since t, P, S, g lie on a circle, the angle $PSt = Pgt = STP$.

25. $QM : PM :: BC : AC :: PN : QN.$

Therefore $QM : CN :: CM : QN.$

Therefore QQ' passes through C.

26. $SY : SP :: BC : CD.$

Therefore $SY.CD = SP.BC.$

27. If T be intersection of tangents at A and B, then, since TC bisects AB, it is a diameter of the conic. Therefore the tangent at C is parallel to AB.

28. The angles SPT, HPt are equal.

Also $TP.Pt = CD^2 = SP.PH,$

or $TP : SP :: HP : Pt.$

Therefore SPT, HPt are similar.

29. $PE = PE' = AC.$

Therefore $SE = HE'$, and the angles SCE, HCE' are equal.

Therefore the circles circumscribing SCE, HCE' are equal.

30. The angles KPG, GPL are equal; therefore KL is a double ordinate of the circle of which PG is diameter.

31. If Q be the centre, QN the ordinate, and T, T' the points where the tangent at P meets the tangents at the vertices,

$$QN^2 : SN.NH :: AT.A'T' : AH.A'S :: BC^2 : A'S^2.$$

(Ex. 20.)

32. Since the tangents are equally inclined to SP, $S'P$ respectively, the bisector of the angle between them bisects SPS', and therefore passes through the point where the axis minor meets the circle.

33. If $PQRS$ be the quadrilateral, p, q, r, s points of contact, H the focus,

the angle $pHP = PHs$, $pHQ = QHq$,
$SHr = SHs$, $rHR = RHq$.

Therefore $PHQ + SHR = PHS + QHR =$ two right angles.

34. $SG : SC :: SP :: SY$ (see Ex. 15)
$:: CD : BC :: PV : VA.$

35. The normals at P and Q will meet on the minor axis in K.

Then angle between the tangents $= PKQ = PSQ$.

36. The auxiliary circle lies entirely without the ellipse except at A and A'; therefore AA' is the greatest diameter.

The circle described on BB' as diameter lies wholly within the ellipse; therefore BB' is the least diameter.

37. Let any circle passing through N and T meet the auxiliary circle in Q.

Then $$CN \cdot CT = CA^2 = CQ^2.$$

Hence CQ touches the circle at Q, and the circle cuts the auxiliary circle orthogonally.

38. The angle $PNY = PSY = PS'Y' = PNY'$.

Therefore $$PY : PY' :: NY : NY'.$$

39. $$PQ^2 : TQ^2 :: SY \cdot S'Y' : TY \cdot TY'.$$

But $$TQ^2 = TY \cdot TY'.$$

Therefore $$PQ^2 = SY \cdot S'Y' = BC^2.$$

Therefore $$PQ = BC.$$

40. If QN and PM be the perpendiculars on the given lines passing through C, R their point of intersection,

$$RN : QN :: CP : CQ;$$

therefore the locus of R is an ellipse of which the outer circle is the auxiliary circle.

41. $$SP : S'P :: SY : S'Y' :: SY^2 : BC^2,$$

and $$S'Q : SQ :: S'Z' : SZ :: BC^2 : SZ^2.$$

Therefore $$SP \cdot S'Q : S'P \cdot SQ :: SY^2 : SZ^2.$$

42. Let Ca, Cb be the conjugate diameters, and Pm, Pn ordinates of P.

Then $$Cm \cdot CM = Ca^2,$$

and $$Cn \cdot CN = Cb^2;$$

$$\therefore CM \cdot Pm : Ca^2 :: Cb^2 : Pn \cdot CN;$$

$\therefore$ the triangle CPM varies inversely as the triangle CPN.

43.

$CAV : CPT :: CA^2 : CT^2 :: CN : CT :: CPN : CPT$; therefore the triangles CAV, CPN are equal.

44. Let TPQ, Ppq be the tangents intersecting the auxiliary circle in P, Q, p, q.

Let E, e be their middle points.

$$PE^2+pe^2=ET^2+Te^2-TP.\,TQ-Tp.\,Tq$$
$$=CT^2+2CA^2-2CT^2=SC^2.$$

Therefore $$PQ^2+pq^2=SS'^2.$$

45. Let QQ' be a diameter equally inclined to the axis with the conjugate to $P'P$.

Then the circles described through P, P', Q and P, P', Q' will touch the ellipse at Q and Q'.

Hence Q, Q' are the points at which PP' subtends the greatest and least angles respectively.

46. Draw the tangent Qr.

Then, since the angles PSQ, QSr are equal, Q always lies on the tangent at the end of the focal chord RSr.

47. The triangle YCY' will be the greatest possible when YCY' is a right angle: P will then lie on the circle of which SS' is diameter.

This intersects the ellipse in four points, provided SS' is greater than BB'.

48. The points where the lines joining the foci of the two ellipses meet the common auxiliary circle are points through which the common tangents pass.

49. The circle passing through the feet of the perpendiculars is the auxiliary circle of the ellipse.

50. Draw QN perpendicular to AB.

Then $$QN : NA :: BP : AP :: CA : AT,$$

and $$QN : BN :: AT : AB.$$

Therefore $$QN^2 : AN.NB :: CA : AB.$$

Therefore the locus of Q is an ellipse of which AB is major axis.

51. PG, GN, NP are at right angles to CD, DR, RC respectively.

Therefore the triangles CDR, PGN are similar.

Therefore $PG : CD :: PN : CR :: BC : AC$.

52. Let PS, QS meet the ellipse and circle again in p, q.

And let $P'Cp'$ be the diameter parallel to SP.

Then, since pq is an ordinate,

$$SQ : SP :: Sq : Sp :: Qq : pP :: AA' : P'p'.$$

Again, $PS.Sp : AS.SA' :: CP'^2 : CA^2$.

Therefore $SR.Pp : 2BC^2 :: P'p'^2 : AA'^2$,

or $Pp : AA' :: P'p'^2 : AA'^2$.

Therefore $SQ : SP :: AA' : Qq$,

and $Qq = P'p'$.

53. If SP meet $S'Y'$ in L', $SL' = AA'$;

therefore $SR = AC$.

54. Since the directions before and after impact are equally inclined to the tangent at the point of impact, the lines in which the ball moves will touch a confocal ellipse or hyperbola.

55. Let the tangent at P meet the tangents at A and A' in F and F'.

Then, since the angles PSF, FSA and PSF', $F'SA'$ are respectively equal, S (and similarly S') lies on the circle of which FF' is diameter.

56. P', D', the two angular points, will lie in PN, DM respectively.

Therefore $P'N : NC :: DM : NC :: BC : AC$.

Therefore P' lies on a fixed straight line through C.

Similarly Q' lies on the other equi-conjugate diameter.

57. The angles SPS', STS' are equal by Ex. 15.

$\therefore SPS' : STS' :: SP.S'P : ST.S'T :: CD^2 : ST^2$.

58. If T be the centre, then, since the angles TSP, TSA are equal,

T lies on the tangent at A.

59. Let QL be the ordinate of Q.

Then $QL : LS :: CN : NP$,

and $QL : LS' :: CM : MD$.

Therefore

$QL^2 : SL.SL' :: CM.CN : PN.DM :: AC^2 : BC^2$,

or Q lies on an ellipse of which SS' is minor axis.

60. This example is the same as No. 12.

61. The tangent at Q is parallel to the normal at P; therefore the tangent at P is parallel to the normal at Q.

62. If $PQ\,P'Q'$ be the parallelogram, the angle DHE is the supplement of

$$HPQ' + HQP';$$

that is, of

$$SPQ + SQP;$$

that is, of

$$DSE.$$

Hence S, H, D, E lie on a circle.

63. Let the line through C parallel to the tangent meet the directrices in Z, Z'.

Since the auxiliary circle is fixed, SY, $S'Y'$ are fixed straight lines meeting ZZ' in fixed points y y'.

And $Cy.CZ = CX.CS = CA^2$.

Therefore Z and Z' are fixed points.

64. The angle $S'TZ = STY = SZY =$ complement of YZT.

Therefore YZ and $S'T$ are at right angles.

65. If G be the centre of the circle, GL bisects SP at right angles.

Therefore SP is equal to the latus rectum.

66. If CZ be perpendicular to YY', the perimeter of the quadrilateral is equal to SS' together with twice $CZ+ZY$, which is greatest when $CZ=ZY$, that is when SPS' is a right angle.

67. Draw SZ perpendicular to $S'Z$ the straight line on which S' lies.

Let $PS'P'$ be the chord parallel to SZ.

Produce PP' both ways to M and M', so that

$$S'M=S'M'=AA'.$$

Then the lines drawn through M and M' perpendicular to SZ are fixed, and $SP=PM$, $SP'=P'M'$.

Hence the ellipse will touch two parabolas having S for focus.

68. Let TQ, TQ' be tangents, V the middle point of QQ'.

Then $$QV.VQ'=CP^2-CV^2=CV.VT.$$

Hence Q, Q', C, T lie on a circle.

69. Draw QM perpendicular to the minor axis.

Then $$QC^2 : AC^2-CN^2 :: BC^2 : AC^2,$$

or $$QC^2 : BC^2 :: AC^2-CN^2 : AC^2.$$

Therefore $$BC^2-CN^2-QN^2 : CN^2 :: BC^2 : AC^2,$$

or $$BC^2-MC^2 : QM^2 :: AC^2+BC^2 : AC^2.$$

Therefore Q lies on an ellipse of which BB' is minor axis.

70. If QM be the ordinate of Q,

$$AM^2 : CN^2 :: QM^2 : PN^2 :: AM.MA' . AC^2-CN^2.$$

Therefore $$AM.AA' : AM^2 :: AC^2 : CN^2,$$

or $$2CN^2=AC.AM.$$

But $$AQ.AO : CP^2 :: AM.AC : CN^2;$$

therefore $$AQ.AO=2CP^2.$$

71. $SP : SN :: SC : CQ :: SC : AC-QR.$

Therefore $SP.AC=SP.QR+SN.SC.$

But $SP : XS+SN :: SC : CA$

or $SP.AC=XS.SC+SN.SC.$

Therefore $SP.QR=XS.SC=BC^2.$

72. If the tangent meet the tangent at A in T, and $S'Y$, $S'Z$ be perpendicular to TS, and the tangent, T, A, Y, S', Z lie on the circle of which $S'T$ is diameter.

The angles YTZ, ATS' are equal since $ATS, S'TZ$ are equal. Art. 68.

Therefore the chords YZ, AS' on which these angles stand are equal.

73. If P, Q, P', Q' be the parallelogram, p, q, p', q' the points of contact, pq, $p'q'$ are parallel focal chords bisected by PCP'.

But QCQ' bisects pp', qq' and is therefore conjugate to PCP' and parallel to pq, $p'q'$.

Therefore $CQ=CQ'=CA$

74. If T be the point from which the tangents are drawn,

ST, $S'T$ are perpendicular to TP', TP respectively.

Therefore SP, $S'P'$ are both parallel to CT.

75. $CS^2 : CA^2 :: CG : CN :: CG.CT : CN.CT.$

Therefore $CG.CT=CS^2.$

76. If PG be the normal at the point of contact,

$$CG.CT=CS^2.$$

Therefore G is a fixed point and P lies on the circle of which GT is diameter.

77. Let the given straight line pq meet the axis t.

Let the tangents at p and q meet in Q.

Let CQ meet pq in V and the curve in P

Through P, Q draw PG, QG' perpendicular to pq, meeting the transverse axis in G and G'.

Then $CG' : CG :: CQ : CP :: CP : CV :: CT : Ct$;

$$\therefore CG' . Ct = CG . CT = CS^2,$$

or G' is a fixed point.

78. Draw SY perpendicular to the tangent; produce SY to L making $LY = YS$.

The point of intersection of the circles described with centres L and P', and radii AA' and $AA' - SP'$ respectively will be the second focus.

79. Draw SY, SY' perpendicular to the given tangents.

The point of intersection of circles described with centres Y and Y', and radius equal to CA will be the centre.

80. If OS be drawn perpendicular to PQ, S will be one focus.

If SP, PS' be equally inclined to OP and SQ, QS' to OQ, S' will be the other.

Bisect SS' in C, and take CA in SS' such that

$$2CA = SP + PS'.$$

If X be the foot of the directrix,

$$CX . CS = CA^2.$$

81. $Qq : Aq :: PN : AN$,

and $Rr : rA' :: PN : NA'$.

Therefore $Qq . Rr : Aq . A'r :: PN^2 : AN . NA'$

$$:: BC^2 : AC^2 :: SL : AC.$$

Now $Aq : qA' :: Aq^2 : Qq^2$,

and $A'r : rA :: A'r^2 : Rr^2$.

Therefore $Aq . A'r : Ar . A'q :: AC^2 : SL^2$.

82. By Ex. 75. $CT : CS :: CS : CG$;

therefore $TS : CS :: SG : CG$.

But $TY : PY :: TS : SG$;

$$\therefore TY^2 : PY^2 :: CS^2 : CG^2 :: CT : CG :: TZ : PZ.$$

83. If O be the intersection of the lines
$$OC^2 = AC^2 + BC^2.$$

84. $$TP : TQ :: CP' : CQ,$$
and the angles PTQ, $P'CQ'$ are equal.
Therefore PQ is parallel to $P'Q'$.

85. S, P, t, S' lie on a circle, and the triangles SCt, PYS are similar.
Therefore $$St : Ct :: SP : SY :: CD : BC,$$
or $$St . PN : Ct . PN :: CD . BC : BC^2.$$
Therefore $$St . PN = CD . BC.$$

86. If the tangent at Q meet the minor axis in t', the angle $SQS' = St'S'$, or t' is on the circle.
Now $$QM . Ct' = BC^2 = PN . Ct.$$
Therefore $$QM : PN :: Ct : Ct' :: Ct : Ct + St$$
$$:: BC : BC + CD \text{ by Ex. 85.}$$

87. If $S'Z$ be perpendicular to TY,
the angle STY = complement of TZY', (Ex. 64.)
= half supplement of YCY' the angle at the centre,
$= CYY'$; and $STY = S'TY'$.

88. The tangents at L and L' are perpendicular to the tangent at P, and therefore D and D' where they meet the tangent at P are on the director circle.
Now $$DL : DP :: D'L' : D'P;$$
therefore PQ bisects the angle LPL'.
Therefore $LP + PL'$ = diameter of director circle.

89. AB, AE are equally inclined to BC,
and $$AB^2 = AD . AE.$$
Therefore AB is a tangent.

90. If the tangent at P meet the tangent at A in T, TS, TS' bisect the angles PSA, $PS'A$.

91. If the chords of intersection NO, PQ meet in T and CD, CE, $C'D'$, $C'E'$ are parallel radii,

$$CD^2 : CE^2 :: TN.TO : TP.TQ :: C'D'^2 : C'E'^2.$$

92. If PN meet CD in K,

$$PK : PQ :: SG : SP :: SA : AX,$$

and $$PN.PK = PG.PF = BC^2.$$

Therefore PQ varies inversely as PN.

93. Draw perpendiculars SY, CE, $S'Y'$, SZ, CF, $S'Z'$ on tangents TP, TQ,

then $$CT^2 = CF^2 + TF^2 = CZ^2 + TZ.TZ' = CA^2 + SY.S'Y' = CA^2 + CB^2.$$

94. If the circle meet the minor axis in K and L, the tangents at P and Q meet either in K or L, see Ex. 15.

95. This problem is equivalent to Ex. 45.

96. Let CV bisecting the chord QSQ' meet the curve in P and directrix in T.

Let DCD' be the parallel diameter.

Then $$SR : SC :: CS - CR : CS :: CT - CV : CT$$
$$:: CS^2 - CG^2 : SC^2 :: SG.GS' : CS^2$$
$$:: SP.PS' : CA^2 :: CD^2 : CA^2 :: QQ'^2 : DD'^2$$

by Art. 76.

97. Let the tangent at Q meet PN in P' and the axis in U.

Then $$CT.CN = CA^2 = CM.CU;$$

therefore $$CT : CM :: CU : CN,$$

or $$TM : CM :: NU : CN.$$

But $$PN.Q'M : Q'M^2 :: TN : TM,$$

or $$PN.Q'M : CM.MU :: CN.NT : CM.NU.$$

Hence $$PN.QM : CN.NT :: CM.QM : CM.P'N$$
$$:: QM^2 : P'N.QM.$$

Now $$PN^2 : CN.NT :: BC^2 : AC^2 :: QM^2 : Q'M^2.$$

Hence $$PN.Q'M : PN^2 :: Q'M^2 : P'N.QM.$$

Therefore $P'N : PN :: AC : BC,$

or P' is on the auxiliary circle.

98. The diameter bisecting PQ is fixed, hence V the centre of the circle, is a fixed point.

VM bisecting RS at right angles, is a fixed straight line;

PQ and RS are equally inclined to the axis;

$\therefore$ CM and CV are equally inclined to the axis.

Therefore M is a fixed point, and RS a fixed straight line.

99. The angle BSC

$$= BAC + SBA + SCA = BAC + HBC + HCB$$
$$= BAC + \text{supplement of } BHC.$$

Hence if BHC is constant, BSC will be constant.

100. The angles SPT, HPt are each equal to SQH;

also $$STP = tQS - PtH$$
$$= HPt - PtH = PHt.$$

Therefore $TP : SP :: HP : Pt,$

or $$TP \cdot Pt = SP \cdot HP = CD^2.$$

Therefore CT, Ct are conjugate.

CHAPTER IV.

THE HYPERBOLA.

1. If the circle whose centre is P touch the circles whose centres are S and H in Q and R,

$$SP \sim HP = SQ \sim HR.$$

Therefore P lies on an hyperbola of which S and H are foci.

2. $SD^2 = BC^2 = CS^2 - CA^2 = CS^2 - CX \,.\, CS = CS \,.\, SX.$

Therefore the triangles SCD, SDX are similar.

3. If the straight line meet the curve in P and the directrix in F,

$$SF : SX :: CS : CA :: SA : AX :: SR : SX.$$

Therefore $SF = SR.$

Draw PK perpendicular to the directrix.

Then $PF : PK :: SC : CA :: SP : PK.$

Therefore $SP = PF.$

4. Draw SD, SD' perpendicular to the asymptotes.

Then DD' is the directrix.

5. If the asymptote meet the directrix in D, then DS drawn at right angles to CD meets the axis in the focus.

6. If PK, QL be the perpendiculars from the given points on the directrix $PS-SQ=PK-QL$ which is constant.

Therefore S lies on an hyperbola of which P and Q are foci.

7. If the circle inscribed in the triangle ABC touch the sides in D, E, F; B, C, D being given,

$$BA-CA=BF-EC=BD-DC.$$

Hence A lies on an hyperbola of which B and C are foci and D a vertex.

8. $PN : Pg :: SY : SP :: BC : CD$
$:: PF : AC :: AC : Pg.$

Therefore $PN=AC.$

9. Draw CD parallel to the given line and CP parallel to the tangent at D.

Then the tangent at P is parallel to CD and the given line.

10. Let $A'P$ and $P'A$ meet in Q, and draw the ordinate QM.

Then $QM : A'M :: PN : NA'$,

and $QM : AM :: P'N : NA.$

Therefore $QM^2 : AM.MA' :: PN^2 : AN.NA'$
$:: BC^2 : AC^2$,

or Q lies on an hyperbola having the same axes.

11. Let the tangent at P, AP and $A'P$ meet the minor axis in t, E and E'.

Then $CE : PN :: CA : AN :: CA.A'N : AN.NA'$,

and $CE' : PN :: CA.AN :: AN.NA'.$

Hence $CE-CE' : PN :: 2CA^2 :: AN.NA'.$

Now $PN^2 : AN.NA' :: PN.Ct : AC^2$;

therefore $CE-CE'=2Ct.$

Therefore Pt bisects every line perpendicular to AA' terminated by $A'P$, AP.

12. SPT is an isosceles triangle since pST is parallel to SP.

Therefore $$SP = ST.$$

13. Draw SD perpendicular to the asymptote and SK parallel to it.

If TS bisect the angle PSK, T being on the asymptote, TP is the tangent at P. Draw SY perpendicular to it.

Then CM which bisects DY at right angles will meet the asymptote in the centre C. DX drawn perpendicular to CS will be directrix.

14. If the tangent at P meet the tangents at A and A' in Z and Z', ZS and $Z'S$ are the internal and external bisectors of the angle ASP.

Hence the foci lie on a circle of which ZZ' is diameter.

15. Draw PK perpendicular to the directrix and DS at right angles to the asymptote.

Draw cxs at right angles to the directrix meeting it in x.

With centre P and radius PS such that $SP : PK :: cs : cD$, describe a circle meeting Ds in S.

Then S is the focus.

16. Draw Qq', Pp and Rr, Qq parallel to the asymptotes.

Then $$CP : CQ :: Cp : Cq' :: Cq : Cr :: CQ : CR.$$

17. $$QC^2 - CB^2 : CN^2 - CA^2 :: BC^2 : AC^2$$
$$:: PN^2 : CN^2 - CA^2.$$

Therefore $$PN^2 = QB \cdot QB'.$$

18. $$DN : NA :: QM : MA \text{ and } EN : NA'$$
$$:: QM : MA'.$$

Therefore $$ND \cdot NE : AN \cdot NA' :: QM^2 : AM \cdot MA'$$
$$:: PN^2 : AN \cdot NA',$$

or $$PN^2 = ND \cdot NE.$$

19. A, A', Y, Z lie on the auxiliary circle.

Therefore $$AT \cdot TA' = YT \cdot TZ.$$

20. If the tangent at P meet the asymptotes in L and L', $CD=PL=PL'$; therefore Q divides LL' in a constant ratio.

Draw QH. QK parallel to the asymptotes.

Then QH. QK varies as CL . CL' and is therefore constant.

Therefore Q lies on an hyperbola having the same asymptotes.

21. This is equivalent to the preceding.

22. Since $SK=S'K$, K lies on the circle passing through S, S' and P, and since KPt is a right angle, t lies on the same circle.

Therefore $GK : S'K :: SG : SP :: SA : AX$,

and $St : tK :: SY : SP :: BC : CD$.

23. Let P, P' be the points of trisection of the arc SS' and let XM bisect SS' at right angles, then $SP=2PM$ and $S'P'=2P'M$.

Hence P and P' lie on hyperbolas of which S and S' are foci and XM directrix.

If C, C' be the centres $CS=4CX$ and $C'S'=4C'X$.

Therefore C and C' are the points of trisection of the chord SS'.

24. Draw SZ parallel to the asymptote: the angle $STQ=TSZ=TSP$.

Therefore $SQ=QT$.

25. Since the hyperbolas have the same asymptotes the ratios $CS : BC : CA$ are constant.

Let NP be the fixed line parallel to an asymptote, and PQ proportional to an axis.

Then PQ^2 varies as CS^2, that is, as $CN.NP$, that is, as NP.

Hence Q lies on a parabola having NP for a diameter.

26. $PY.PY' = AC^2 - CP^2 = CD^2 - BC^2 = CS^2$
$$- \left(\frac{SP - S'P}{2}\right)^2 = CS^2 - CA^2 = CB^2.$$

27. Let TP meet the other asymptote in T', then $PT = PT'$.

Therefore $PQ = R'P = QR$.

28 Draw OrR parallel to PQ, meeting the ellipse and hyperbola in r and R.

Let Oa, Ob be the axes, then since OP, Or are conjugate in the ellipse, and OQ, OR in the hyperbola, if PN, QM, rl, RL be the ordinates,

$$ON.Ob = rl.Oa;\ PN.Oa = Ol.Ob;\ QM.Oa = OL.Ob;\ OM.Ob = RL.Oa.$$

Therefore $PN : ON :: QM : OM$

since $rl : Ol :: RL : OL$,

or OP and OQ are equally inclined to the axes.

29. Through S draw SC parallel to the bisector of the angle between the asymptotes meeting the asymptote which is given in position in C.

Draw SD perpendicular to that asymptote, and DX to CS.

Then if A be taken in CS such that $CA^2 = CX.CS$, A is vertex.

30. If the tangents at P and Q meet in T, then since the perpendiculars from T on SP, SQ, HP, HQ are all equal, a circle can be described with centre T to touch SP, SQ, HP and HQ.

31. If CL, CL' be the asymptotes, S will lie in the bisector of the angle LCL'.

Draw PL, PL' parallel to the asymptotes to meet them in L and L';

and take S in CS such that $CS^2 = 4CL.CL'$,

then S is a focus.

32. If the conjugate diameters PCP', DCD' be given, complete the parallelogram $LML'M'$ formed by the tangents at D, P, D' and P'.

The diagonals LL', MM' are the asymptotes and the axes bisect the angles LCM, LCM'.

33. Let QT and RQ meet the asymptotes in L and M.

Then $QL : PH :: RH : TL :: CR : CT$

$:: RM : TK :: PK : QM$;

therefore $QL \cdot QM = PH \cdot PK$,

or Q is on the curve.

34. Let CL bisecting the angle ACB' meet PN in L, draw QM parallel to LC.

Then CL is proportional to CN;

therefore $CL \cdot CM$ is proportional to $CN \cdot NT + CN^2$, that is to CA^2.

Hence Q lies on an hyperbola of which CL and CB are asymptotes.

35. Draw SY perpendicular to the tangent and produce it to Z making $YZ = SY$.

Then if Q be the point of contact, and P the fixed point,

$$HP - PS = HQ - QS = HZ;$$

therefore $$HP - HZ = PS,$$

or the locus of H is an hyperbola of which P and Z are foci.

36. If PT, Pt the tangents to the ellipse and hyperbola meet BC in T and t, then since the curves have the same conjugate axis, for $CA^2 = CS^2 + CB^2$

$$Ct \cdot PN = BC^2 = CT \cdot PN,$$

or $$CT = Ct.$$

37. This problem is the converse of Ex. 3.

38. If G be the point of intersection

$$CG = \tfrac{2}{3} CP,$$

or G lies on an hyperbola having the same asymptotes.

39. The angle $CYY' = S'PY' = S'NY'$ since S', P, N and Y' lie on a circle.

Therefore Y, Y', C and N lie on a circle.

40. If SY meet $S'P$ in Z, $SY = YZ$; therefore $S'Y$ bisects PG.

Similarly SY' bisects PG.

41. $BC^2 : AC^2 :: NG : CN :: CT.NG : CN.CT$;

therefore $$CT.NG = BC^2.$$

42. The angle $STP = TS'P + S'PT = TS'P + SPT$

$= PS'S + SS't =$ supplement of PSt.

43. $$Pt.PT = CD^2 = SP.S'P,$$

or $$SP : PT :: Pt : S'P;$$

and the angles SPT and tPS' being equal, the triangles SPT, tPS' are similar.

44. The circles SCE, $S'CE'$ stand upon equal chords SC, $S'C$ and contain equal angles SEC, $S'E'C$, since CE is parallel to the bisector of SPS'.

45. If the tangent at P meet the tangent at A, the vertex of the branch on which P lies, in T, T is the centre of the circle inscribed in the triangle SPS', since TS, TS' bisect the angles ASP, $AS'P$.

46. $CT : CA :: CA : CN :: CP : CQ$, or AQ is parallel to PT.

47. $$CE : CA :: CS : CA;$$

therefore $$CE = CS;$$ but $CD = CA.$

Therefore AD and SE are parallel.

48. If E be the centre of the circle and EK its radius,

$$EK : CE :: BC : SC,$$

or $$EK : CA :: BC : SC+BC$$
$$:: (SC-BC)\,BC : CA^2.$$

And $$SR' : CS :: BC : AC :: SR : BC;$$

therefore $$RR' : SR :: CS-BC : BC,$$

or $$RR' : CS-BC :: SR : BC :: BC : CA.$$

Therefore $$EK=RR'.$$

49. $$PM=PL;$$

therefore $$GL=GM.$$

50. If Ca, Cb be the conjugate diameters and one hyperbola touch the ellipse in P, the tangent at P will meet Ca, Cb in T, t, such that $TP=Pt=CD$.

Hence PD is bisected by Ct,

and tD touches the other hyperbola and is parallel to CP.

51. If LL' and MM' be the tangents,

$$CL : CM :: CM' : CL',$$

or LM' and $L'M$ are parallel.

52. If the tangent at P meet the tangents at A and A' in F and F' and QM be the ordinate of the centre of the circle,

$$QM : MS :: SA : AF,$$

and $$QM : MS' :: S'A' : A'F'.$$

Hence $QM^2 : SM.MS' :: SA^2 : AF.A'F' :: SA^2 : BC^2$

(Art. 126).

Hence the locus of Q is an hyperbola of which S and S' are vertices.

53. If PM be perpendicular to the directrix,

$$PK : PM :: CS : CA :: SA : AX :: SP : PM,$$

or $$PK=SP.$$

54. Let PD meet an asymptote in n, draw Pl, Dm parallel to Cn.

Then $$Dm \cdot Dn = Pn \cdot Pl;$$

therefore $$Dn = Pn.$$

Therefore if LPL' is tangent at P, LD is tangent at D, and CP, CD are conjugate.

55. If QR meet the asymptotes in q and r, $qQ = rR$; therefore if EPe be the tangent at P,

$$CL : CN :: qQ : Pe :: PE : qR :: CN : CM.$$

56. If the circle intersect the axis in b, B,

$$CB \cdot Cb = CS^2$$

or $$CB^2 + CB \cdot Bb = CA^2 + CB^2;$$

therefore $$CB \cdot Bb = CA^2.$$

57. Let the straight line $q'Q'APQq$ meet the asymptotes in Q', Q.

Draw RCR' parallel to AP terminated by $A'q$, $A'q'$.

Then $$PQ' = AQ = CR = Qq,$$

and $$Q'q' = CR' = AQ' = PQ;$$

therefore $$Pq' = Pq.$$

58. T is the centre of the circle inscribed in the triangle $PS'Q$, therefore the difference between PTQ and half $PS'Q$ is a right angle.

59. Draw CD, CE parallel to OA, OB and PH, PK parallel to and terminated by CE, CD.

Then $$PH : OD :: CK : CE :: CP : CA :: CB : CP$$
$$:: CD : CH :: OE : PK;$$

therefore $$PH \cdot PK = OD \cdot OE,$$

or P lies on an hyperbola having CD, CE for asymptotes.

60. Draw PH, PK, QH', QK' parallel to the asymptotes.

Then $$PL : QM :: PH : QH' :: QK' : PK :: QN : PR,$$

or $$PL \cdot PR = QM \cdot QN.$$

61. If TK, TN be perpendicular to the directrix and SP, $TK = SN$.

Therefore $ST : TK :: ST : SN$ a constant ratio,

and the angle between the asymptotes is double PST, that is, double the external angle between the tangents.

62. $$Q'V^2 - RV^2 = CD^2 = RV^2 - QV^2,$$

or $$QV^2 + Q'V^2 = 2RV^2.$$

Again $$CT.CV = CP^2 = CV.CT';$$

hence $$CT = CT'$$

63. If V be the middle point of PQ, then since R, V are the middle points of LRL' and $LPQl$, RV is parallel to the asymptote $CL'l$.

Hence $$PM + QN = 2RE.$$

64. If TP, TQ be the tangents, PTQ, STS' have the same bisector which passes through the point where the circle meets BCB'.

65. The tangents at P and Q intersect in t on the circle and BCB'.

Hence the angle $PtQ = PSQ$.

66. The angle $PNY = PSY = PS'Y' =$ supplement of PNY'.

67. The triangle YCY' is greatest when YCY' or SPS' is a right angle.

In that case PT meets BC in t such that $Ct = CS$;

therefore $$CS.PN = Ct.PN = BC^2.$$

68. The triangle SPS' : triangle $StS' :: SP.PS' : St^2 :: CD^2 : St^2$.

69. $S'P$ is parallel to CY and $S'Q$ to CZ.

Therefore $S'T$ is parallel to bisector of YCZ and is perpendicular to YZ.

70. If G be the centre of the circle, GL bisects SP;

therefore $$SP = 2PL = 2SR.$$

71. The tangent at Q is parallel to the normal at P, therefore the tangent at P is parallel to the normal at Q, or CP is conjugate to normal at Q.

72. If Y be the point from which the tangents are drawn, SP and $S'P'$ are both parallel to CY.

73. $SC^2 : AC^2 :: CG : CN :: CG.CT : CN.CT$,
or $CG.CT = SC^2$.

74. By Ex. 73, G the foot of the normal is a fixed point;

therefore P lies on the circle of which TG is diameter.

75. If TP, TQ be the tangents, CT will bisect PQ in V,
and $CT.CV = CT^2$,
or PQ is a tangent at V.

76. Let GQ meet the conjugate in G'.
Then $QG' : QG :: CN : NG :: AC^2 : BC^2$.
Therefore, by Art. 111, QG' is normal at Q.

77. If PM be drawn perpendicular to the directrix of the parabola the angle $PTQ = SPT - SQT =$ half $SPM -$ half $SQS' =$ half $SS'Q$.

78. If $abcd$ be the quadrilateral and S lie on the circle the angle $Hcd = Scb = Sab = Had$,
or H is on the circle.

79. If PP' be the chord of contact and CV bisect PP' then CV, PP' are parallel to a pair of conjugate diameters in both conics.

Hence if from a common point Q, a double ordinate QVQ' be drawn parallel to PP', Q' must lie on both curves.

Similarly RR' the line joining the other two common points is parallel to PP'.

80. If SD, SD' are perpendiculars from the common focus on the asymptotes, DD' is the tangent at the vertex of P and a directrix of H.

If P be a common point, and PM perpendicular to DD',

$$SP : PM :: SC : CA,$$

but $$SP = PM + SX.$$

Therefore $SP : SX :: CS : CS - CA :: CS : AS.$

Hence $$AS . SP = SX . CS = BC^2 = AS . SA',$$

or $$SP = A'S.$$

Therefore $A'P$ touches the parabola at P.

81. With centre P, the given point and radius of the given length describe a circle meeting the other asymptote in p.

Then $pPQq$ is the line required.

82. Let CB, CA be semiaxes of the ellipse, Ca, Cb of the hyperbola.

Let PN meet the asymptote in Q,

then $$QN^2 : CN^2 :: Cb^2 : Ca^2,$$

or $$QN^2 + CN^2 : Ca^2 + Cb^2 :: CN^2 : Ca^2;$$

but $$SP + S'P = 2CA,$$

and $$SP - S'P = 2Ca.$$

Hence $4CA . Ca = SP^2 - S'P^2 = SN^2 - S'N^2 = 4CN . CS;$

therefore $$CA^2 : CS^2 :: CN^2 : Ca^2 :: QN^2 + CN^2 : Ca^2 + Cb^2 :: CQ^2 : CS^2.$$

Therefore Q lies on the auxiliary circle of the ellipse.

83. Let Q be a common point.

Then $SQ - QH = AA'$ and $SQ - QP = SP - 2PH = AA' - PH.$

Therefore $$QP = QH + PH,$$

or Q must be the other extremity of the focal chord PH.

84. If $A'K$ meet the directrix in F, then,

since $SA' = 2A'X$,

$FA'S$ is an isosceles triangle and FS is parallel to KD.

Also $A'F : FP :: A'X : XN :: A'S : SP$

or FS bisects the angle $A'SP$;

therefore if SP and DK meet in Q, QSD is an isosceles triangle.

Therefore Q lies on the circle of which $A'D$ is diameter.

85. This problem is a particular case of Ex. 61.

86. $PL \cdot PL' = PL^2 = CD^2 = PG \cdot Pg$;

therefore G, g, L, L' lie on a circle of which Gg is diameter.

C is on this circle since GCg is a right angle.

The radius of this circle varies as Gg and therefore as CD and therefore inversely as the perpendicular from C on LL'.

87. If PCP', DCD' be conjugate diameters and Q any point on the curve,

$QP^2 + QP'^2 = 2CP^2 + 2CQ^2$; $QD^2 + QD'^2 = 2CD^2 + 2CQ^2$.

Therefore

$QP^2 + QP'^2 - QD^2 - QD'^2 = 2CP^2 - 2CD^2 = 2AC^2 - 2BC^2$.

88. If $S'L'$, $S'M'$ be drawn parallel to the asymptotes LS', MS' bisect the angles $PS'L'$, $PS'M'$.

Hence $LS'M$ = half the angle between the asymptotes.

89. If PT meets the tangent at A in V, VS bisects the angle ASP;

therefore $SP : ST :: PV : VT :: AN : AT$.

90. This problem is the same as Ex. 24.

91. Let $ABCD$ be the quadrilateral, A, B, and C being fixed points.

Then $AB+CD=BC+AD,$

or $CD-DA=CB-AB.$

Hence D lies on an hyperbola of which A and C are foci.

92. Since Q, S, C, t lie on a circle, the angle

$$tQC=tSS'=SPt,$$

hence CQ is parallel to SY and CY, SQ are equally inclined to SY;

therefore $SQ=CY=CA.$

93. Draw SY perpendicular to the tangent and produce to Z making $SY=YZ.$

Then if P be the point of contact $HZ=HP-SP='AA'.$

Hence the locus of H is a circle of which Z is centre.

94. RS and VS' bisect the angles PSQ and $PS'Q$; let QS, $S'P$ meet in Z.

Then $RSP+VS'Q=\text{half } PSQ+\text{half } PS'Q=\text{half } QSP$

$+\text{half } SZS'-\text{half } SQS'=QSP+\text{half } SPS'-\text{half } SQS'$

$$=QTP+TQS-SQT=PTQ.$$

95. CgP is an isosceles triangle, and the angle

$$CGt=CPT=TCP;$$

therefore $PG=Ct$ and $CD^2=PG.Pg=Cg.Ct=CS^2.$

96. Since the asymptote CD bisects BA, CD is parallel to the axis of the parabola and BA is parallel to the other asymptote.

If $QPVP'Q'$ parallel to BA meet CD in V,

$$QV=VQ' \text{ and } PV=VP';$$

therefore $QP=Q'P'.$

97. Let EL be the ordinate of E, and draw EF perpendicular to PN.

Then, CD being conjugate to CP, the triangles CDM and PFE are similar and equal.

$$\therefore CL = CN + EF = CN + DM,$$
$$\therefore CN : CL :: AC : AC + BC,$$

and similarly $EL : PN :: BC : BC - AC$;

$$\therefore EL^2 : (BC - AC)^2 :: PN^2 : BC^2$$
$$:: CN^2 - AC^2 : AC^2$$
$$:: CL^2 - (AC + BC)^2 : (AC + BC)^2.$$

98. If PM be drawn from the centre perpendicular to BC

$AP : PM :: PC : PM$, a constant ratio;

therefore P lies on an hyperbola of which A is focus and BC directrix.

If S be the other focus and SP meet the circle in Q

$$SQ = SP - PA = \text{constant},$$

or, the envelope is a circle of which S is centre.

99. The conics will be confocal having their foci H and H' on PG,

such that $$PH^2 = PT . Pt = CD^2.$$

For their locus see Ex. 9 on the ellipse.

100. If $SY, SZ, S'Y', S'Z'$ be perpendiculars on tangents at right angles

$$CT^2 - CA^2 = TY . TY' = SZ . SZ' = CB'^2.$$

If $SYZ, S'Y'Z'$ are perpendicular to parallel tangents and CWW' be the perpendicular through the centre

$$2CW = SY + S'Y'; \; 2CW' = SZ - S'Z',$$

and $$SY - SZ = S'Y' + S'Z';$$

$$\therefore 4CW'^2 + 4CB^2 = 4CW^2 - 4CB^2.$$

Hence $$CW^2 - CW'^2 = CB^2 + CB'^2.$$

CHAPTER V.

THE RECTANGULAR HYPERBOLA.

1. The angle $PCL = CLP =$ complement of LCY.

2. $QV^2 = VP \cdot Vp$, hence VQ touches at Q the circle QPp.

3. $$LP = PM = CD = PG = Pg.$$

Hence LGM is a right angle.

4. If LM be the straight line, and C be the corner of the square, $CL \cdot CM$ is constant, hence LM touches an hyperbola of which CL, CM are asymptotes.

5. Let AP, $A'P'$ meet in Q, and draw the ordinate QM.

Then $$QM : MA :: PN : NA,$$

and $$QM : MA' :: P'N : NA'.$$

Hence $$QM^2 : AM \cdot MA' :: PN^2 : AN \cdot NA';$$

therefore $$QM^2 = AM \cdot MA'.$$

Hence Q lies on a rectangular hyperbola having AA' for transverse axis.

6. If P, P' be joined to Q meeting an asymptote in R and R',

the angle

$$QRL = CLP - QPL = LCP - PP'Q = CR'P' = QR'L.$$

7. Produce LP to M making $PM=PL$, then if MC be drawn perpendicular to the given asymptote CL, C is the centre. In CS the bisector of the angle LCM take S such that CS is a mean proportional between CL and CM.

Then S is focus, and X the middle point of CS is the foot of the directrix.

8. The angle $DCL=PCL$,

and $D'CL=P'CL$;

hence $DCD'=PCP'$.

9. A diameter is a mean proportional between the parallel focal chord and AA', therefore focal chords parallel to conjugate diameters are equal.

10. As in the preceding, focal chords at right angles are equal, since diameters at right angles are equal.

11. If CD, Cd be conjugate to CP in the ellipse and hyperbola,

$$CD^2=SP.PS'=Cd^2=CP^2.$$

12. $PN=CM$; $DM=CN$;

and $CD=CP$.

13. $SP.PS'=CD^2=CP^2$.

14. $QV^2=CV^2-CP^2=CV^2-CT.CV=CV.VT$.

Hence QV touches the circle CTQ.

15. If D be the intersection of tangents at A and B,

$$CD^2=AC^2+BC^2=SC^2.$$

Hence D lies on the circle of which SS' is diameter.

16. If LPM, $G'PG$ be the tangent and normal to one,

$$LP=PM=CD=CP=PG=PG';$$

therefore GG' is the tangent to the second hyperbola.

17. The angle $CRT=CQT+RTQ=2CLQ+LTL'$

$$=CLM+CL'T=CLR'+L'CQ'=TR'Q'.$$

Hence C, T, R and R' lie on a circle.

18. $QR^2 = CN^2 = CA^2 + RN^2 = CA^2 + CQ^2 = AQ^2$.

19. Let AQ, AR be the fixed straight lines, and P the middle point of QOR.

Through C the middle point of AO draw CH, CK parallel to AQ, AR,

and through P draw PHM, PKN parallel to AR, AQ.

Then the complements AC, HK about the diagonal MN are equal.

Therefore $PH \cdot PK$ is constant, and P lies on a rectangular hyperbola, having CH and CK for asymptotes.

20. Draw QB perpendicular to AB, and make AB equal to CD, then A is a fixed point.

Then $AD : AB :: BC : CD :: QB : DP$,

or $PD \cdot DA = AB \cdot BQ$.

Hence P lies on a rectangular hyperbola of which AB is one asymptote.

21. If D, E, F, O be the centres of the escribed and inscribed circles,

$$OC \cdot CF = DC \cdot CE,$$

since the triangles OCE, DCF are similar.

Hence the hyperbola is rectangular since diameters at right angles are equal.

22. If the diameters of the parallelogram $LML'M'$ meet in C, the angle $SLS' = SL'S'$.

Hence S and S' lie on a rectangular hyperbola circumscribing $LML'M'$. (Art. 137.)

23. If PSq, SQq be the chords and D be a point on the directrix, such that DS bisects the angle QSp, then D will lie in pq.

But DS is perpendicular to the asymptote since Pp, Qq are equally inclined to it, therefore D lies on the asymptote.

Similarly Pq, Qp meet in D', the foot of the perpendicular from S on the other asymptote.

24. If V be the middle point of PQ and POP' be a diameter,

the angle $VNP = NPV = QP'P = POV$.

Hence O, P, V, N lie on a circle.

If OQ meet the given tangent in T, produce OQ to V, making OV a third proportional to OT and OQ; with centre V and radius, a mean proportional between VO and VT, describe a circle meeting the given tangent in P its point of contact. In the tangent measure off

$$PL = PM = OP,$$

then OL and OM are the asymptotes.

25. Draw PD perpendicular to the base QR,

then, since $PD^2 = DQ \,.\, DR$,

DP is the tangent at P.

26. Let a circle on DE meet the hyperbola in P and Q, draw the diameters PCP', QCQ'.

Then, since the angle DPE is either equal or supplementary to $DP'E$ and DQE to $DQ'E$, the similar circle on the other side of DE, will meet the curve in P' and Q'.

27. Let OAD, OBC be the fixed straight lines, PM, PN, PL perpendiculars from the centre of the circle on BC, AD and the bisector of the angle AOB.

Let PM, PN meet OL in m, n.

Draw Ll, Ll', Pr, Pr' perpendicular to BC, AD, Ll, Ll' respectively.

Then, $BM^2 + MP^2 = AN^2 + NP^2$,

or $PN^2 - PM^2 = BM^2 - AN^2$

which is constant,

$PN^2 = (Ll' + Lr')^2 = (Ll - Lr)^2 + 4Ll' \,.\, Lr' = PM^2 + 4Ll \,.\, Lr$.

Hence the rectangle $Ll \,.\, Lr$ is constant;

but $Ll : OL$ in a constant ratio and $Lr : PL$ is constant.

Therefore $PL \,.\, LO$ is constant, or P lies on a rectangular hyperbola having OL for an asymptote.

28. If P be the point of contact

$$CL=2PN,\ CL'=2CN;$$

therefore $CL.CL'=2Ca^2=4PN.CN.$

Hence $CN.CL' : Ca^2 :: Ca^2 : PN.CL,$

or $AC^2 : Ca^2 :: Ca^2 : CB^2.$

29. Draw CV conjugate to PQ.

Then, the angle

$$TPQ=PP'Q=PCV=CPQ';$$

therefore the angles CPQ, TPQ' are equal.

30. Let DB, DC meet the asymptotes in b and c: draw AH, AK parallel to the asymptotes. Then OBb, OAK are similar triangles, also OCc, OAK.

Hence

$$OB : OA :: Ob : OK :: OH : Oc :: AK : Oc :: OA : OC;$$

therefore A lies on the circle of which BC is diameter as does D.

31. Let the tangents at P and Q meet the asymptote in L and M.

The angle

$$PCQ=PCL-QCM=PLC-CMT=LTM$$
$$=\text{supplement of } PTQ.$$

32. Each hyperbola passes through the orthocentre of the triangle ABC.

Hence D is that orthocentre.

Now the line joining the middle point of AB to the middle point of CD is a diameter of the nine point circle.

And $AB^2+CD^2=$square on diameter of circumscribed circle.

Hence the circles intersect at right angles.

33. $PN^2=CN^2-CA^2=CN^2-CN.CT=CN.NT.$

Hence the triangle CPN is similar to PTN, and therefore to tTC.

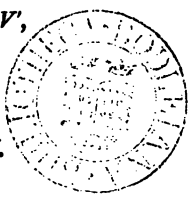
34. This problem is a particular case of Ex. 61 on the hyperbola.

35. CM, CN which bisect PP', PQ' are conjugate being equally inclined to the asymptotes;
therefore $P'Q'$ is a diameter.

36. If AB be a diameter of the hyperbola, CD subtends, at A and B, angles which are both equal and supplementary, and are therefore right angles.

37. If AQ', BQ meet in R, the angle QAQ' = supplement of $QBQ' = RBP$;
therefore R lies on the circle.

38. Let CV, CV' bisect PQ, $P'Q$.
The angle $PRQ = PQL = VCQ = CQV'$,
so $P'R'Q = CQV$.
Draw VM perpendicular to CQ,
then $PQ : QR :: VM : CV$,
and $P'Q : R'Q :: VM : QV$.
Now $P'Q = 2CV$,
and $PQ = 2QV$;
therefore $QR = QR'$.



39. Let PN meet CF in K,
then CF varies as CG, and PF varies as PK, or Ct, or CT.
Hence $PF.FC$ is proportional to CG, CT or CS^2.
Hence P lies on a rectangular hyperbola having CF for asymptote.

CHAPTER VI.

THE CYLINDER AND THE CONE.

1. Take two points E and A on the generating line, and draw EX at right angles to the axis, making EX equal to EA.

Then the plane containing AX, and perpendicular to the plane EAX will cut the cylinder in an ellipse of the required eccentricity.

2. Take two points E and A on the generating line and with centre A, and radius twice EA, describe a circle meeting EF in X.

Then the plane through AX perpendicular to the plane EXA will intersect the cone in an ellipse of the required eccentricity.

3. Take two points EA on the generating line, the least angle of the cone will be, when EA is double the perpendicular from A on EF, that is when the semi-vertical angle is equal to the angle of an equilateral triangle.

4. A tangent plane to a cone touches it along a generating line OF, hence OF is parallel to all sections parallel to the tangent plane which are therefore parabolas.

If C be the centre of the sphere FES, the ratios $CS : CA$ and $CA : CO$ are constant, and the angle OCS is constant, therefore COS is constant, and S lies on a cone of which O is vertex and OC axis.

5. Through the flames of the candles which are treated as points, draw planes intersecting in the ceiling, in a straight line, since these fixed planes must always be tangent planes to the ball, the locus of the centre of the ball is a horizontal straight line.

6. The triangles AEX, $A'E'X'$ have all their sides parallel;

therefore

$$SA : AX :: EA : AX :: E'A' : A'X' :: S'A' : A'X'.$$

7. If C be the centre of the sphere FES, the angle OCS is constant.

And the ratios $CE : CA$, $CE : CV$, and $CS : CA$ are all constant;

therefore $CS : CV$ is constant and the angle CVS is constant, or VS is a fixed straight line.

8. Take two points E and A on the generating line and with centre A and radius AX, such that $EA : AX$ in the ratio of the eccentricity, describe a circle intersecting FE in X, the section of which AX is axis will have the required eccentricity.

9. XS, XS' are tangents to the same sphere FES of which C is centre.

Let SS', EF meet the axis in V', L, and let CX meet SS' in M.

Then $$CL \cdot CV' = CM \cdot CX = CE^2.$$

Hence V and V' coincide.

10. Draw CN perpendicular to the axis,

then $$2CN = A'D' - AD,$$

and $$2ON = OD + OD'.$$

In CN take a point Q, such that

$$QN : CN :: DO^2 : DA^2;$$

then since $OD' - OD : 2CN :: OD : DA$,

$2QN : OD' - OD :: DO : DA$;

also $AD + A'D' : 2ON :: AD : DO$,

and $AA'^2 = DD'^2 + (AD + A'D')^2$.

Hence $QO^2 : CA^2 :: OD^2 : DA^2$.

Hence Q lies on a sphere of which O is centre, and therefore C lies on a spheroid, having OD for its axis, which is oblate or prolate according as DO is greater or less than DA, that is according as the vertical angle of the cone is greater or less than a right angle.

11. Draw a plane CT through C, the centre of the sphere perpendicular to the axis intersecting the tangent plane at S in T. Then since the angles COE and CTS are equal, and $CE = CS$, it follows that $CT = CO$.

Hence S lies on the surface generated by the revolution of the circle of which CT is diameter.

12. Only two circles can be described passing through S, and touching the generating lines in which the plane through S and the axis intersects the cone.

If ST, ST' be the tangents, and OS meet the circles in D and D',

the angle $DST = \text{half } SCD = \text{half } SC'D' = D'ST'$.

Hence the planes of the corresponding sections make equal angles with OS.

13. If the plane of section meet the plane through O perpendicular to the axis in the line ZZ', and PK be perpendicular to ZZ'

OP or OQ bears to PK a constant ratio.

Hence if P' in the projection corresponds to P

$OP : P'K$ in a constant ratio.

But OP' is equal to the perpendicular from P on the axis which bears to OP a constant ratio.

Hence the ratio $OP' : P'K$ is constant.

14. $A'Q - QA = SS'$ and a right cone can be constructed of which Q is vertex, such that the generating lines intersect the ellipse.

Hence

$$PQ + AS = AS + QR + RP = AE + EQ + SP = AQ + SP.$$

15. Through the vertical straight line which is the locus of the luminous point draw vertical tangent planes to the ball intersecting the inclined plane in OY, OZ. The locus of C the centre of the shadow will be the straight line bisecting YZ, at right angles, SY, SZ being perpendiculars from the point of contact of the ball on OY, OZ which are tangents to the elliptic shadow.

16. The given plane to which the sections are perpendicular must be supposed to contain the axis of the cone.

Take OK on the generating line equal to AS.

Draw KG perpendicular to OK, meeting a line through O at right angles to the axis in G, then G is a fixed point.

Draw GM perpendicular to AS,

then $\quad CE : EO :: OK : KG,$

or $\quad KG : KA :: AE : CE.$

Hence the angle $KAG = ACE =$ half KAM.

Hence AS touches a circle centre G and radius GK.

17. $VP = VQ = VA + AQ = 2AE + AN = 2AS + AN.$

18. EX', $X'A'$ will be parallel to EX, XA.

Hence if AF be drawn parallel to $A'E'$, the ratio of the eccentricities is $AE : AF$ which is constant.

19. The volume varies as the area AVA' and BC; and the area AVA' varies as $AV \,.\, VA'$, or $AD \,.\, A'D'$, that is, BC^2.

Hence BC is constant.

20. $A'O - OA = A'E' - EA = A'S - SA = SS'$.

Hence the locus of O is an hyperbola of which A and A' are foci.

21. $BC^2 = EC \cdot CE'$.

Hence the locus of C is an hyperboloid of revolution generated by the revolution round the axis of the cone of an hyperbola of which OE, OE' are asymptotes.

22. CA and EA are the bisectors of the angle OAS.

Hence the sphere on CE as diameter, intersects the plane of section in a circle of which AA' is diameter.

CHAPTER VII.

CURVATURE.

1. SL being a fourth of the chord of curvature at L through S, LG the normal is a fourth of the diameter of curvature.

2. The normal and tangent at L are equally inclined to the axis.

3. Draw SO at right angles to SP, meeting the normal at P in O, and from O draw OQ at right angles to AP; then the chord of curvature through A is $4PQ$; drawing AZ perpendicular to TP,

$$PQ : OP :: AZ : AP,$$

or $$PQ . AP = OP . AZ.$$

Again, $$SP : PO :: AZ : AT,$$

or $$OP . AZ = SP . AT;$$

also $$PY : SP :: AT : TY,$$

or $$PY^2 = SP . AT = PQ . AP.$$

Hence $$4PQ : 4PY :: PY : AP.$$

4. The diameter at P and SP are equally inclined to the normal at P; hence chord of curvature parallel to the axis is equal to $4SP$.

5. Let QM be the ordinate of Q.

Then $$QM : MG :: PN : NG,$$

or $$PN.MG = 2AS.QM = pn.Mg.$$

Also $$Nn = Gg.$$

Hence $$MG : Mg :: pn : PN,$$

or $$MG : Gg :: pn : pn - PN.$$

Therefore $$MG : pn :: Nn : pn - PN :: pn^2 - PN^2 : 4AS\,(pn - PN) :: pn + PN : 4AS.$$

But $$QM : MG :: PN : 2AS;$$

hence $$QM : 2PN :: pn\,(pn + PN) : 16AS^2.$$

If P and p approach each other indefinitely, Q is the centre of curvature,

and $$PQ : PN + QM :: PG : PN.$$

But $$QM : PN :: PN^2 : 4AS^2 :: AN : AS;$$

hence $$QM + PN : PN :: AN + AS : AS :: SP : AS.$$

Hence $$PQ : PG :: SP : AS :: PG^2 : SR^2.$$

6. If PK be perpendicular to the directrix and S' the farther focus,

$$SP : PK :: CA : CX;$$

hence $$SP = \tfrac{1}{2}AC,$$

and $$S'P = \tfrac{3}{2}AC.$$

Therefore $$PE.PS' = \tfrac{3}{2}AC^2 = 2SP.S'P = 2CD^2,$$

or the circle of curvature passes through S'.

7. If PV be the chord of curvature through the focus and pp' the focal chord parallel to the tangent at P,

$$PV.AC = 2CD^2,$$

or $$PV.AA' = DD^2 = AA'.pp';$$

hence $$PV = pp'.$$

8. Let Q be the middle point of the chord and QM its ordinate.

If PNP' be the double ordinate, $P'Q$ is a diameter.

If PQ meet the axis in W,

$$WM = TN = NW;$$

hence $$AM = 5AN,$$

and $$QM^2 = PN^2 = 4AS \cdot AN = \tfrac{4}{5} AS \cdot AM;$$

or Q lies on a parabola of which A is vertex and AS axis.

Produce SA to S' making $S'A = 3AS$, and let PQ meet the tangent at A in Y'.

Then $AY'^2 = \frac{9}{4} PN^2 = 9AS \cdot AN = 3S'A \cdot AN = S'A \cdot AW$.

Hence $S'Y'$ is perpendicular to PQ, and PQ envelopes a parabola of which S' is focus and A vertex.

9. DR and PCP' are equally inclined to the axis, and D, R, P, P' lie on a circle;

hence PR, DP' are equally inclined to the axis.

So DQ, PD' are equally inclined to the axis;

hence PR, DQ are parallel, since DP', PD' are parallel.

10. $$PG = CD = CP.$$

Hence the radius of curvature varies as CP^3.

11. $$LP \cdot PC = PT \cdot Pt = CD^2.$$

Hence PL is equal to half the chord of curvature in direction PC.

$$CP \cdot CL = CP \cdot PL + CP^2 = CD^2 + CP^2 = AC^2 + BC^2.$$

12. The circle on PE as diameter touches the curve at P and goes through Q; when Q coincides with P the circle becomes the circle of curvature.

13. If the tangents at two near points P and Q meet in T,

$$TP : TQ :: CD : CE.$$

Hence the difference of TP and TQ is very small compared with either, and if a circle be described, of which the intersection of normals at P and Q is centre, to touch TP and TQ at P and Q, when P and Q coincide this becomes the circle of curvature at P.

14. If C be the centre of curvature at the vertex,

$$AC = 2AS.$$

If PR be the tangent,

$$PR^2 = CP^2 - CR^2 = PN^2 + CN^2 - 4AS^2$$
$$= 2AC \cdot AN + CN^2 - AC^2 = AN^2.$$

15. PCP' and the tangent at P are equally inclined to the axis.

16. If O be the centre of curvature,

$$PO^2 = AC \cdot C = PF \cdot CD.$$

But $$PO \cdot PF = CD^2,$$

hence $$PO = CD = PF.$$

Hence, if with centre C and radius CP, such that

$$CP^2 = AC^2 + BC^2 - AC \cdot BC,$$

a circle be described, it will meet the curve in points, at which the radius of curvature has the required value.

17. If PV be the chord and CQ be drawn parallel to Pt,

$$2CD^2 = PQ \cdot PV = Ct \cdot PV.$$

But $$Ct \cdot PM = BC^2,$$

hence $$PV : PM :: 2CD^2 : BC^2.$$

18. If PQ be the common chord of the ellipse and circle of curvature, TPt must make equal angles with both axes, or $CT = Ct$.

Make the angle ACP such that

$$PN : NC :: BC^2 : AC^2;$$

then CP will meet the ellipse in the point required.

19. SP is one-fourth of the chord of curvature through S, hence PQ is half the radius of curvature.

20. If PV be the chord,

$$PV \cdot PD = 2CD^2.$$

But $$PN = ND,$$

hence $$PV : CD :: CD : PN.$$

21. $$PG : PF :: PK : PC :: PE : Pg,$$

or $$PF \cdot PE = PG \cdot Pg = CD^2;$$

hence E is the centre of curvature.

22. Draw SQ at right angles to SP to meet the normal in Q; let the tangent at P meet the directrix in Z.

Then $$PL : PS :: ZP : ZS :: PQ : PS.$$

Therefore $PL = PQ =$ half the radius of curvature.

23. If CE bisect PQ, PCE is a right angle and

$$PF \cdot PE = CP^2 = CD^2;$$

hence PQ is equal to the diameter of curvature at P.

24. $$OP : CP :: PQ : PF,$$

or $$OP \cdot PF = CP^2 = CD^2;$$

hence O is the centre of curvature at P.

25. If the normal at P meet BC in K; S, P, H, K lie on a circle; hence, when P coincides with B, K becomes the centre of curvature at B.

26. If HT be produced to H' making $H'T = TH$, TQ, PH' will be parallel;

hence $$PR : RS :: H'T : TS :: TH : TS.$$

Therefore $$PR : PS :: TH : 2TC :: HP : 2CY :: HP : 2CA,$$

or $$2PR \cdot AC = PS \cdot HP = CD^2.$$

Hence PR is one-fourth of the chord of curvature through S.

27. If O be the centre of curvature at A,

$$SO : SA :: SA : AX.$$

Hence curvature of ellipse is greater than that of parabola, and curvature of parabola is greater than that of hyperbola.

28. By Ex. 6, $SP = \frac{3}{2}AC$,

and $$SP : CY' :: PE : PC.$$

Hence $$PE : EP' :: 3 : 1.$$

29. Let CQ' be conjugate to CF and CP' parallel to PQ.

Then $$OE^2 : OF^2 :: OQ.OP : OF^2 :: CP^2 : CQ^2$$
$$:: CD^2 : CQ^2 :: TP^2 : TF^2 :: TE^2 : TF^2.$$

Hence $TEOF$ is cut harmonically.

CHAPTER VIII.

PROJECTIONS.

1. THE theorems are obtained by projection from the following properties of the circle.

Art. (65) Every diameter of a circle is bisected at the centre, and the tangents at its extremities are parallel.

(70) If the chord of contact of tangents from T to a circle meet CT in N, $CT \cdot CN = CA^2$.

(71) If M, t correspond to M', t' on the circle,

$$CM : CM' :: BC : AC :: Ct : Ct';$$

therefore $$CM \cdot Ct = BC^2.$$

(73) If the chord of contact of tangents from T to a circle meet CT in V and CT meet the curve in P,

$$CT \cdot CV = CP^2.$$

(74) A diameter of a circle bisects all chords parallel to the tangents at its extremities.

(75) If a diameter of a circle bisects chords parallel to a second, the second diameter bisects all chords parallel to the first.

(76) Art. 77 is meant.

If PCP', DCD' be diameters of a circle at right angles,

$$QV^2 : PV \cdot VP' :: CD^2 : CP^2.$$

(78) If CA, CB be radii at right angles,

$$CN^2 = AM \cdot MA',\ CM^2 = AN \cdot NA',$$
$$CM : PN :: AC : BC :: CN : DM.$$

(81) The area of a square circumscribing a circle is constant and equal to the rectangle contained by diameters at right angles.

(82) If CD, CP be radii at right angles and the tangent at P meet a pair of radii at right angles in T and t,

$$PT \cdot Pt = CD^2.$$

(83) Cor. 1. The two tangents TP, TQ from any point are equal, and the parallel diameters ACA', BCB' are equal,

$$\therefore TP : ACA' :: TQ : BCB;$$

and these ratios are unaltered by projection.

Cor. 2. In the circle TCT' is a right angle,

and $$\therefore PT \cdot PT' = CP^2 = CD^2,$$

CD being parallel to TPT'.

(84) Taking ABA' as a semicircle, ECF is a right angle; project on any plane parallel to the line ACA'.

2. If a parallelogram be inscribed in a circle its sides are at right angles. The greatest rectangle than can be inscribed in a circle is a square having its area equal to $2AC^2$; hence the greatest parallelogram that can be inscribed in an ellipse has its area equal to $2AC \cdot BC$.

3. The theorem is true in the case of a circle, and follows by projection.

4. The greatest triangle which can be inscribed in a circle is an equilateral triangle of which C is the centre of gravity.

Produce PC to V, making $2CV = PC$;

then, if QVQ' be the ordinate, PQQ' is the greatest triangle which can be inscribed in the ellipse having its vertex at P.

5. If a straight line meet two concentric circles, the portions intercepted between the curves are equal.

6. The locus of the point of intersection of tangents at the extremities of diameters of a circle at right angles is a concentric circle.

7. The locus of the middle points of lines joining the extremities of diameters of a circle at right angles is a concentric circle.

8. If CP, CD be radii of a circle at right angles and CA bisect the angle PCD, the tangent at A meets CP in T such that $PD^2 = 2AT^2$.

9. If a chord AQ of a circle be produced to meet the diameter at right angles to CA in O and CP be parallel to AQ,

$$AQ \cdot AO = 2CP^2.$$

10. If OQ, OQ' are tangents to a circle and R be a diagonal of the parallelogram of which OQ, OQ' are adjacent sides, then if R be on the circle the locus of O is a concentric circle.

11. If a parallelogram be inscribed in a circle and from any point on the circle straight lines are drawn parallel to the sides of the parallelogram, the rectangles under the segments of these lines made by the sides are equal to one another.

12. If a square circumscribe a circle and a second square be formed by joining the points where its diagonals meet the circle, the area of the inner square is half that of the outer. And if four circles be inscribed in the spaces between the outer square and the circle, their centres will lie on a concentric circle.

13. If a rectangle be inscribed in a circle so that the diameter bisecting one pair of sides is divided in a constant ratio, the area is constant.

14. If a parallelogram circumscribe a circle and one of its diagonals bear a constant ratio to the diameter it contains, the area is constant.

CHAPTER IX.

CONICS IN GENERAL.

1. $TN : XN :: SR : SX :: SP : XN$;

hence $SP = TN$.

Also $TP \cdot TP' = TN^2 - PN^2 = SP^2 - PN^2 = SN^2$.

2. Draw Pm, Qn perpendicular to the directrix.

Then $PR : QN :: KP : KQ :: Pm : Qn :: SP : SQ :: PM : QN$,

or $PR = PM$.

3. $PS \cdot SQ : AS \cdot SA' :: Cp^2 : CA^2$,

and $PQ \cdot SR = 2SP \cdot SQ$.

Hence PQ varies as Cp^2.

4. Let Pp, Qq intersect in O.

Then $QO \cdot Oq : PO \cdot Op :: TP^2 : TQ^2 :: QO^2 : PO^2$.

Hence TO bisects pq as well as PQ, and is a diameter and goes through t.

5. Since RS is the exterior bisector of the angle $P'SQ'$,

$$SP' : SQ' :: RP' : RQ'.$$

6. Let $S'T$, ST meet PQ in EE', and let TF, TG be the common interior and exterior bisectors of the angles ETE', PTQ.

Bisect FP in O.

Then $OE.OE' = OF^2 = OP.OQ$.

Now $RP : RQ :: PE : EQ$,

and $R'P : R'Q :: PE' : E'Q$.

Again, $OP : OE :: OE' : OQ$;

hence $PE : OE :: E'Q : OQ$.

Also $OP : OE' :: OE : OQ$;

hence $PE' : OE' :: EQ : OQ$.

Therefore $RP.PR' : RQ.QR' :: PE.PE' : EQ.QE'$
$:: OE.OE' : OQ^2 :: OF^2 : OQ^2$.

Hence TF bisects the angle RTR', and the angles RTP, $R'TQ$ are equal.

7. TS and KS are the interior and exterior bisectors of the angle PSP'.

Hence if ST meet PP' in E,

$RK : TE :: KP : EP :: KP' : EP' :: KR' : ET$,

or $RK = KR'$.

8. If DE, $D'E'$ are perpendicular to SP, SP',

then $SE = SE'$.

Hence DE, $D'E'$ intersect on the bisector of the angle PSP', which is ST.

9. If PP' meet the directrix in K, PP' is harmonically divided at S and K.

Hence any chord through S is harmonically divided by the directrix and the tangents at P and P'.

10. Draw SY perpendicular to the tangent, then since

$SC : CY :: SA : AX$,

the locus of C is a circle.

CHAPTER IX.

CONICS IN GENERAL.

1. $TN : XN :: SR : SX :: SP : XN$;

hence $SP = TN$.

Also $TP \cdot TP' = TN^2 - PN^2 = SP^2 - PN^2 = SN^2$.

2. Draw Pm, Qn perpendicular to the directrix.

Then $PR : QN :: KP : KQ :: Pm : Qn :: SP : SQ :: PM : QN$,

or $PR = PM$.

3. $PS \cdot SQ : AS \cdot SA' :: Cp^2 : CA^2$,

and $PQ \cdot SR = 2SP \cdot SQ$.

Hence PQ varies as Cp^2.

4. Let Pp, Qq intersect in O.

Then $QO \cdot Oq : PO \cdot Op :: TP^2 : TQ^2 :: QO^2 : PO^2$.

Hence TO bisects pq as well as PQ, and is a diameter and goes through t.

5. Since RS is the exterior bisector of the angle $P'SQ'$,

$$SP' : SQ' :: RP' : RQ'.$$

6. Let $S'T$, ST meet PQ in EE', and let TF, TG be the common interior and exterior bisectors of the angles ETE', PTQ.

Bisect FP in O.

Then $OE.OE' = OF^2 = OP.OQ.$

Now $RP : RQ :: PE : EQ,$

and $R'P : R'Q :: PE' : E'Q.$

Again, $OP : OE :: OE' : OQ;$

hence $PE : OE :: E'Q : OQ.$

Also $OP : OE' :: OE : OQ;$

hence $PE' : OE' :: EQ : OQ.$

Therefore $RP.PR' : RQ.QR' :: PE.PE' : EQ.QE'$
$:: OE.OE' : OQ^2 :: OF^2 : OQ^2.$

Hence TF bisects the angle RTR', and the angles RTP, $R'TQ$ are equal.

7. TS and KS are the interior and exterior bisectors of the angle PSP'.

Hence if ST meet PP' in E,

$RK : TE :: KP : EP :: KP' : EP' :: KR' : ET,$

or $RK = KR'.$

8. If DE, $D'E'$ are perpendicular to SP, SP',

then $SE = SE'.$

Hence DE, $D'E'$ intersect on the bisector of the angle PSP', which is ST.

9. If PP' meet the directrix in K, PP' is harmonically divided at S and K.

Hence any chord through S is harmonically divided by the directrix and the tangents at P and P'.

10. Draw SY perpendicular to the tangent, then since

$SC : CY :: SA : AX,$

the locus of C is a circle.

11. Let Pp meet the curve in P', and let QP' meet the directrix in q'.

Then since pS, $q'S$ are the exterior bisectors of the angles PSP', QSP', the angle pSq' = half PSQ = pSq; hence q and q' coincide.

12. $SL : SP :: FT : FP :: TN : PK.$

Therefore $SL : TN :: SP : PK :: SA : AX.$

13. $PN^2 : AC^2 - CN^2 :: CB^2 : CA^2 :: Cb^2 : Ca^2$

$$:: pn^2 : Cn^2 - Ca^2.$$

Hence, if $PN = pn$, $AC^2 - CN^2 = Cn^2 - Ca^2$,

or $$CN^2 + Cn^2 = CA^2 + Ca^2.$$

14. $QR : LG :: PQ : PG :: PM : PN.$

Hence $QR : PM :: LG : PN :: SG : SP :: SA : AX.$

15. Draw KV parallel to the axis.

Then $VK : VP :: GS : SP :: SG' : SQ :: VK : VQ,$

or $$PV = VQ.$$

Also $PL : PM :: PK : PG :: PV : PS;$

hence $2PL . PS = PM . PQ = SR . PQ = 2SP . SQ,$

or $$PL = SQ.$$

Hence $SV = VL$, and the diagonal of the parallelogram SL goes through V.

16. If PQR be the triangle and S the focus, make the angles QSr, QSp each equal to the supplement of PSR.

Then, if $PSq = PSr$, p, q and r are the points of contact.

17. By Ex. 20, Chap. I., if KV be drawn parallel to the axis, $PV = VQ$.

Hence $PN : PL :: PK : PG :: PV : PS.$

Hence $2PN . PS = SR . PQ = 2SP . SQ,$

or $$PN = SQ.$$

Hence $SN = 2SV$, and the locus of N is a similar conic.

18. Let CT, CT' meet the curve in p, d.

Then $$CT \cdot PR = 2Cp^2,$$

and $$CT' \cdot QR = 2Cd^2.$$

Hence the triangle

$$CTT' : 2 \text{ triangle } Cpd :: 2Cpd : PRQ,$$

or the triangle

$$CTT' : AC \cdot BC :: AC \cdot BC : \text{the triangle } PRQ.$$

19. Let S be the centre of the circumscribed circle, H the ortho-centre,

then the feet of the perpendiculars from S and H on AB, BC, CA lie on the nine-point circle,

and the angle SAB = complement of $C = HAC$.

Therefore with S and H as foci a conic can be inscribed in ABC.

20. Let SV meet the directrix in Q and PK in Z, let QP meet the axis in C.

Then $$PZ : SP :: SG : SP :: SA : AX;$$

hence $$PZ : PK :: SA^2 : AX^2.$$

Now $$SC : CX :: PZ : PK :: SA^2 : AX^2,$$

or C is the centre.

21. $$DE \cdot DF : DC^2 :: AB^2 : AC^2 :: DG^2 : DC^2,$$

or $$DG^2 = DE \cdot DF.$$

22. By Ex. 74 on the parabola, if PGQ be the chord of contact,

$$DF : FG :: PG : GQ :: GF : FE,$$

or $$FG^2 = FD \cdot FE.$$

23. If Epq be drawn parallel to DTP,

$$DP^2 : Ep \cdot Eq :: DF : EF.$$

For DF is parallel to a generating line VM of the cone of which the hyperbola is a section.

Draw Dlm, LEM in the plane $VFEM$ to meet the cone.

Then the sections of the cone by the parallel planes, lPm, pLQ are similar,

and $$Dm = EM.$$

Hence

$$DP^2 : Ep \cdot Eq :: Dl \cdot Dm : EL \cdot EM :: Dl : EL :: DF : EF.$$

Again, $Ep \cdot Eq : EQ^2 :: PT^2 : TQ^2 :: EK^2 : EQ^2$;

hence $EK^2 = Ep \cdot Eq$ if Epq meet PQ in K.

And $DG^2 : GE^2 :: DP^2 : EK^2 :: DP^2 : Ep \cdot Eq :: DF : EF.$

Therefore $$FG^2 = FD \cdot FE.$$

24. Let the tangents at P, Q and R meet EB in p, q and r.

Then $Ep \cdot Eq = EF^2$, if PQ meet EB in F;

also $EB^2 = Er \cdot Ep$, $EC^2 = Er \cdot Eq$.

Hence $EB^2 : EC^2 :: Ep : Eq$, a constant ratio.

By Ex. 23. the same proposition is true in the case of an hyperbola if EB be parallel to an asymptote.

25. GK being perpendicular to SP,

$$Pk : PK :: Pg : PG :: AC^2 : BC^2;$$

$$\therefore Pk \text{ is constant.}$$

Also $$kL : Pk :: SG : SP;$$

$$\therefore kl \text{ is constant.}$$

CHAPTER X.

HARMONICS, POLES AND POLARS.

1. If AOB be the common chord and $PQOpq$ any transversal,

$$PO \cdot Op = AO \cdot OB = QO \cdot Oq.$$

2. OA is perpendicular to $B'C'$ and meets it in D,

$OA \cdot OD$ = square on radius of concentric circle $= OB \cdot OE$
$= OC \cdot OF.$

Therefore $OD = OE = OF.$

3. Draw ABC to be bisected by OB in B, BEG to AO produced to be bisected by OC in E, and BFK to CO produced to be bisected by OA in F.

Then any one of the straight lines drawn through O parallel to AC, BG, BK will form a harmonic pencil with OA, OB, OC.

Draw BL, BM parallel to OA, OC to meet OC, OA in L and M.

Then since $OE = EL$, $OF = FM$, EF is parallel to LM and is therefore bisected by OB and is also parallel to ABC.

Hence the pencil BC, BE, BO, BF is harmonic.

4. Let the circles meet in P, bisect AC in E.

Then $EB \cdot ED = EC^2 = EP^2$,

hence the circles cut at right angles.

5. By Art 182, PQ, AE, BD intersect in A, and $A\{B, E, Q, F\}$ is harmonic.

6. AP, BQ and PB, AQ meet each pair on the polar of O on which C lies;

And the pencil formed by CB, CO, CA and the polar of O is harmonic.

7. If A, B be points of contact and the third conic meet AB in C and D, A and B are the foci of the involution, P, Q are conjugate points.

Hence $ACBD$ is a harmonic range.

8. Let the common chords meet in E, and let $EPRQ$ be a tangent at R; then since the common chords are one conic of the system, E and R are foci of the involution $EPRQ$ and $EPRQ$ is a harmonic range.

9. Let TP, TQ be the tangents, TE any line, then F the pole of TE lies on PQ and $PEQF$ is a harmonic range.

10. If the tangent at P meet the asymptotes in L and L', $PL = PL'$.

Hence the pencil CD, CL, CP, CL' is harmonic.

11. A diameter is bisected at the centre: and the polars of the extremities of a diameter intersect at infinity.

12. If T be the pole of QR and H the second focus of the conic which touches the ellipse at Q,

$$PQ + QH = SQ + QS';$$

or
$$HS' = SP.$$

Therefore
$$HS' + S'P = S'P + SP;$$

or S' is on the conic of which H is focus.

Again the angles $TS'R$, $TS'Q$ are equal, and PS', $S'H$ are equally inclined to TS'; hence TS' is a tangent at S.

Therefore T lies on the directrix of the conic of which P, H are foci.

13. Let ST, $S'T$ meet PQ in E, E'; then the angles ETP, $E'TQ$ are equal.

Now the ranges $RPEQ$, $R'QE'P$ are harmonic and QTP is common to the two pencils; hence the angles $R'TQ$, RTP are equal.

14. The middle points of all chords of the cone parallel to the given line, lie in a plane through the vertex, let this plane meet the given line in P and any section through it in A and A'.

Then Q the pole of the given line lies in $AA'P$ and $AQA'P$ is a harmonic range. Since VA, VA' are fixed generating lines, VQ is a fixed straight line.

15. If Tpq be the chord, P its pole, then PN the ordinate of P is the polar of T.

Let CP meet pq in v and the curve in Q.

Let QM, QG' be the ordinate and normal at Q.

If PG be drawn perpendicular to pq, it is parallel to QG';

hence $CG : CG' :: CP : CQ :: CN : CM$;

Therefore $CG : CN :: CG' : CM :: SC^2 : AC^2$.

Hence G is a fixed point.

16. Let the polar of Q meet the conjugate CFD in R. Draw QQ' parallel to CP.

Then $PE : PQ :: RF : PF$;

and $PG : PQ :: CF : PF$;

hence $EG : PQ :: CR : PF$;

or $EG . PF = PQ . CR$.

Now Q is on the polar of R, since R is on the polar of Q.

Hence $PQ . CR = CQ' . CR = CD^2$.

Hence EG is equal to the radius of curvature at P.

17. If O be the orthocentre of ABC and $A'B'C'$ the reciprocal triangle, $B'C'$, $C'A'$, $A'B'$ are perpendicular to OA, OB, OC respectively, and BC, CA, AB are perpendicular to OA', OB', OC' respectively.

Hence ABC and $A'B'C'$ have their sides parallel and O is the orthocentre of each.

18. If $pPSQq$ be the focal chord and the tangents at P and Q meet in T, TS is perpendicular to PQ, hence the tangents at p and q meet in T.

19. This theorem is the reciprocal of the following: if two circles intersect they have two common tangents: if one circle lie entirely within the other, they have no common tangents. Reciprocate with respect to a point on one circle and within the other.

20. If I, C be the centres of the inscribed and circumscribed circles, and CI meet them in r, r' and B, B' respectively, then if AA' be the major axis of the ellipse into which the circumscribed circle is reciprocated,

$$IA \,.\, IB = Ir^2, \quad IA' \,.\, IB' = Ir^2,$$

$$CI^2 = CB^2 - 2CB \,.\, Ir.$$

Hence $IA : Ir :: Ir : CB - CI :: CB + CI : 2CB.$

and $IA' : Ir :: Ir : CB + CI :: CB - CI : 2CB.$

Hence $AA' : Ir :: 2CB : 2CB;$

or $$AA' = Ir.$$

21. The four circles which circumscribe the triangles of a complete quadrilateral meet in a point.

22. See Ex. 30 on the parabola, or by reciprocation.

23. See Ex. 46 on the ellipse, or by reciprocation.

24. Let CPP', COO' be perpendicular to the polars of P and O.

Draw OX, OA perpendicular to CP and the polar of P;

PY, PB perpendicular to CO and the polar of O.

Then $CO' : CP' :: CP : CO :: CY : CX.$

Hence $CO' - CY : CP' - CX :: CY : CX :: CP : CO$

or $$PB : OA :: CP : CO.$$

25. See Ex. 18 on Chapter I.

26. (1) If a quadrilateral circumscribe a conic a pair of opposite sides subtend at the focus angles which are together equal to two right angles.

(2) If we reciprocate with respect to the focus S the new theorem is, if Q be taken on a circle and QL be drawn such that the angle SQL is constant, QL envelopes a conic of which S is focus.

27. The envelope of chords of a circle which subtend a constant angle at a fixed point on the circle is a smaller concentric circle.

28. Two circles, such that a point can lie within both cannot have more than two common tangents.

But if the circles be such that all points lie without both, or within one and without the other they may have four common tangents.

29. If a straight line meet the sides of the triangle $A'B'C'$ in L, M, N the circles circumscribing the triangles $A'B'C'$, $A'NM$, $B'NL$, $C'LM$ meet in a point.

30. If points P', Q' be taken on a circle of which C is the centre, $P'C$ will meet the line drawn through Q' at right angles to $P'Q'$ and $Q'C$ will meet the line drawn through P' at right angles to $P'Q'$ on the circle.

31. If S be the orthocentre of the triangle ABC and circles be described with centres A and B passing through C, S will lie on the radical axis of the two circles.

If we reciprocate with respect to S we see that if with the orthocentre of a triangle as focus we describe two conics each touching a side of the triangle and having the other two sides as directrices, the conics will have a parallel pair of common tangents and therefore their minor axes equal.

32. If a system of circles have two points in common the locus of their centres is a fixed straight line, and the polar of a fixed point meets the radical axis in a fixed point.

33. If the tangent and normal at P meet QR in T and G, the range $TRGQ$ is harmonic, since TP, PG bisect the angle QPR.

Hence PG is the polar of T.

Hence the pole of QR lies on PG since the pole of PG lies on QR.

34. If the tangents at P and Q meet in T and TA meet PQ in L, the range $DPLQ$ is harmonic; hence the pencil TD, TP, TL, TQ and the range $DBAC$ are harmonic.

Therefore $ABDC$ is a harmonic range.

35. If the pencil joining $BPAQ$ to any point on the curve is harmonic, the pencil formed by joining them to any other point on the conic is harmonic.

For if BK, PK, AK, QK meet the directrix in $bpaq$, $bpaq$ is a harmonic range, provided $KEBPAQ$ be a harmonic pencil.

And the angles bSp, pSa, aSq, qSb are half the angles BSP, PSA, ASQ, QSB;

Hence the pencil Sb, Sp, Sa, Sq is the same wherever K be taken on the curve.

Now PQ goes through O the pole of AB: let PQ meet AB in R.

Then if T be the pole of PQ, $TARB$ is a harmonic range.

Therefore the pencil joining Q to $BPAQ$ is harmonic;

hence the pencil joining q to $BPAQ$ is harmonic.

Hence Pq bisects AB since AB, qQ are parallel.

36. Four circles can be described so as to touch the sides of a triangle, and the reciprocal of the radius of the inscribed circle is equal to the sum of the reciprocals of the radii of the other three.

If the triangle be equilateral the inscribed circle touches the three escribed circles.

37. If the tangents at P and Q meet the axes in T and V; the angle

$$PSQ = SQV - SPT = SVQ - STP = VST.$$

If SW be perpendicular to $P'Q'$, the tangents at the vertices intersect in W.

Draw SYZ perpendicular to the tangents at P and Q.

Then WSP', WSQ' are supplementary to WYP', WZQ'.

Hence $P'SQ'$, PSQ are supplementary.

If two circles intersect in P, Q the angle between the tangent at P, Q is equal to the angles which the centres subtend at S and supplementary to the angle which PQ subtends at the other point of intersection.

38 and 39. If from any point P in the radical axis tangents be drawn to the circles, and a circle be described, with centre P and radius equal to the tangent, this circle will intersect the line of centres in two points E and F which are the limiting points of the system.

Take A at centre of one of the circles, and M at the point where the radical axis intersects the line of centres.

Then, PU and PU' being the tangents from P to the circle,

$$PM^2 + ME^2 = PE^2 = PU^2 = PA^2 - AU^2;$$

$$\therefore ME^2 = AM^2 - AU^2 = MF^2;$$

$$\therefore AE \cdot AF = AM^2 - EM^2 = AU^2.$$

$\therefore$ the polar of F passes through E.

Reciprocating with regard to F, the pole of uU', i.e. of the fixed line through E, is the centre, which is therefore fixed, and the conics are confocal.

Therefore, if we reciprocate with regard to either limiting point we obtain confocal conics.

40. If perpendiculars be drawn from A, B, C to BC, CA, AB these lines will meet in a point O, and the circles circumscribing ABC, OBC, OCA, OAB are equal.

41. If the tangents at P and Q, points on a circle intersect at a constant angle, and lines be drawn through

P and Q making constant angles with the tangents at P and Q respectively, this pair of straight lines will intersect on a concentric circle.

42. If two circles intersect in A and B and PQ be a common tangent and QB, PA meet the circles in C and D, then PC, QD are parallel.

43. If from any point on a circle circumscribing a triangle perpendiculars be drawn to the sides of the triangle, the feet of these perpendiculars lie on a straight line.

44. Since the orthocentre is on the hyperbola, DEF is a self-conjugate triangle and the pole of EF lies on BC.

Hence the pole of BC lies on EF.

45. If AB, CD meet in the fixed point E, CA and BD in F, and BC and AD in G, then FG is the polar of E.

Hence the centre of the circle lies in a straight line through E perpendicular to FG the polar of E with respect to both curves.

46. The radius of an escribed circle of an equilateral triangle is $\frac{3}{2}$ the radius of the circumscribed circle, and if SE be the tangent from the centre of the circumscribed circle to the escribed circle whose centre is D;

$$SD = DE + \tfrac{1}{3}DE = \tfrac{4}{3}DE.$$

The proposition in the question is obtained by reciprocating with respect to the circumscribed circle.

47. If AD be drawn parallel to the axis to meet BC, AD is bisected at D' where it meets the curve.

Hence the tangent at D' is parallel to BC and bisects AB and AC.

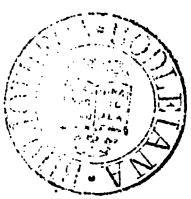
Since a straight line intersects a conic in two points and two tangents can be drawn from a point, the reciprocal polar of a conic with respect to another conic is a third conic.

Now by Ex. 44, if a rectangular hyperbola circumscribe a triangle DEF it will go through the ortho-centre O and ABC the triangle formed by joining the feet of the perpendiculars is a self-conjugate triangle, and O is the centre of the circle inscribed in ABC. If we reciprocate with respect to O the reciprocal conic is a parabola, since it has one tangent at an infinite distance and ABC is a self-conjugate triangle.

The tangents from O are at right angles, since the hyperbola was rectangular, hence O is on the directrix.

The locus of the poles of the lines at an infinite distance, that is, of the centres of the hyperbolas, was the circle circumscribing ABC.

Hence the envelope of the polars of O with respect to the parabolas is an ellipse inscribed in ABC having O for a focus. Since O is now the centre of the circle circumscribing ABC, the auxiliary circle of the ellipse is the nine point circle.



MISCELLANEOUS PROBLEMS.

1. $$2SP \cdot SP' = SR \cdot PP',$$

and $$2SQ \cdot SQ' = SR \cdot QQ'.$$

2. If OPP', OQQ' are the chords and pSp', qSq' parallel focal chords,

$$OP \cdot OP' : OQ \cdot OQ' :: pS \cdot Sp' : qS \cdot Sq' :: pp' : qq'.$$

3. $$QN : CM :: BC : AC :: DM : CN,$$

hence $$QN + DM : NM :: BC : AC.$$

4. If $CVED$ be conjugate to PQ,

$$PQ \cdot Qp = PV^2 - QV^2.$$

Hence $$PQ \cdot Qp : CD^2 - CE^2 :: CR^2 : CD^2,$$

CR being parallel to PQ.

5. $$QN : NX :: SA : AX :: SR : SX.$$

Hence Q lies on the tangent at the extremity of the latus rectum.

6. $$PN^2 : AN \cdot NA' :: BC^2 : AC^2$$

and $$QN \cdot PN = AN \cdot NA'.$$

Therefore $$QN^2 : AN \cdot NA' :: AC^2 : BC^2.$$

7. Let AP, QB meet in R, and draw RV parallel to POQ.

Then $RV : VA :: PO : AO$,

and $RV : VB = QO : OB$.

Hence $RV^2 = VA \cdot VB$,

since $AO \cdot OB = PO \cdot OQ$.

Therefore QR lies on a concentric rectangular hyperbola.

8. The line joining T to the intersection of the normals at P and P' bisects PP' and therefore passes through the centre.

9. If the tangent at P meet the tangents at A and A' in T and T' and TS, $T'S'$ meet in Q,

the angle $SS'Q = T'S'A' = T'S'P$;

$QSS' = AST = TSP$;

and $SPT = S'PT'$.

Hence S, P, S' are the feet of the perpendiculars of the triangle TQT'.

Therefore QP is perpendicular to TP.

10. Make the angle PSF a right angle, then the tangent at P meets the directrix in F: if a circle be described with centre P and radius PK such that the ratio $SP : PK$ is equal to the eccentricity the directrix is a tangent from F to this circle.

Two tangents can in general be drawn.

If the angle SPF be such that $SP : PF :: SA : AX$, only one conic can be constructed; there are two positions of PF equally inclined to SP corresponding to this case.

If the eccentricity be unity, one conic is a line parabola through S.

11. If PK be drawn perpendicular to the directrix of the parabola $SP = PK$,

hence $HM = HP + PK = AA'$.

Therefore the directrix touches a circle of which H is centre.

12. Draw RN perpendicular to the minor axis.
Then
$$CN.AC = SR.AC = BC^2 = AC^2 - SC^2 = AC^2 - RN^2;$$
Hence
$$RN^2 = AC\{AC - CN\},$$
or R lies on a parabola.

13. If ST, PQ meet in H, $HTRS$ is a harmonic range.

But OH, OT, OV, OS is a harmonic pencil.

Hence OV passes through R.

14. Let ACA' be the diameter bisecting the parallel chords QN, etc. in N, etc.

Then PN^2 varies as QN^2, that is as $AN.NA'$.

Hence the locus of P is an ellipse. The locus will be a circle if $PN = QN$, that is if the vertical angle is a right angle.

15. If PP', QQ' be the double ordinates of the given points, P, P', Q, Q' are fixed points, and since the ellipses are similar, the corresponding points of the auxiliary circle, at which the major axis subtends a right angle, are likewise fixed points.

16. The ordinates of the point and of the end of one of the radii are in the ratio of the radii.

17. If P be the centre of the circle and PK perpendicular to the fixed straight line, the ratio $SP : PK$ is constant.

18. Let pqr be a triangle touching the parabola in P, Q, R.

Parabolic area $PQR = \frac{2}{3}$ triangle PqR.

$\therefore$ Triangle $PQR = \frac{2}{3}(PqR - PrQ - QpR)$,

$$3PQR = 2(pqr + PQR);$$

$$\therefore PQR = 2pqr.$$

19. If a rectangular hyperbola circumscribe a triangle the orthocentre is on the curve, if we reciprocate with respect to the orthocentre we have the case of a parabola inscribed in a triangle the tangents from the orthocentre being at right angles. See Ex. 47, Ch. 10.

20. Produce HA to K making AK equal to AH, then PK and AL are parallel.

Hence $SQ : SP :: SA : SK :: SA : SA+AH$.

Therefore the locus of Q is a similar ellipse of which S is focus.

21. If $KLMN$ be the quadrilateral and k, l, m, n the points of contact, KM will bisect nk, lm: and LN will bisect kl, mn.

Hence $klmn$, and therefore $KLMN$, is a parallelogram.

22. The locus of the second focus is a circle of which the radius $= AA - SP$.

The locus of the centre which bisects SH is similar, that is, a circle.

23. Since LC, LL' are tangents, the angles HLC and SLL' are equal.

Again $CL \cdot CL' = CH^2$,

hence the angle $CHL = CL'H = SL'L$.

Therefore $CL : HL :: SL : LL'$,

the triangles CLH, SLL' being similar.

24. If the theorem be true in the case of a circle, it will follow by orthogonal projection for any ellipse.

If PQ be the chord of contact of tangents drawn to a circle from a point on a concentric circle, the angles PAQ, $PA'Q$ will be constant, A, A' being extremities of a fixed diameter.

Let AP, $A'Q$ meet in R, and $A'P$ and AQ in R'.

The angle $ARA' = APA' - PA'Q$,

and the angle $ARA' = APA' + PAQ$.

Hence the loci of R and R' are circles passing through A and A'.

25. Circumscribe circles to two of the triangles formed by the intersections of the tangents, these circles intersect in the focus S: the pedal line of S is the tangent at the vertex.

A parabola can be drawn to touch five straight lines, if the circles circumscribing the triangles formed as above all meet in the same point S.

26. PF is the same for both curves, and therefore CD is also the same.

27. Prove that $SY \cdot S'Y'$ is constant.

28. By reciprocation.

29. P, Q, R, R' lie on a circle of which PQ is diameter and PQ, LL' are equally inclined to the axis. If p, p' are the vertices of the diameters bisecting PQ, RR' in V and V', pp' is a double ordinate.

Let VV' which is parallel to the normal at p' meet the axis in O.

Let VM, $V'M'$ be the ordinates of V and V'.

Then $LL' = L'M' + M'O + MO - LM = 2OM = 2ng = 4AS$.

30. The bisectors are tangent and normal to a confocal conic.

Hence $$CG \cdot CT = CS^2.$$

31. Reciprocate the following theorem: if S, A, B, C be points on a circle and with centres A, B, C and radii AS, BS, CS circles are described, they will intersect two by two in points which lie in a straight line.

32. $OE : EG = SP : SG = Sp : Sg = OE : Eg$.

33. If an ellipse be reciprocated with respect to its centre, the reciprocal is a similar ellipse having its major axis in the minor axis of the original ellipse.

If we reciprocate an ellipse circumscribing a triangle and having its centre at the orthocentre with respect to that orthocentre, the reciprocal is a similar ellipse, inscribed in a triangle having its sides parallel to those of the original triangle, the homologous axes being at right angles, and having its centre at the orthocentre.

This reciprocal ellipse is similar and similarly situated to the ellipse inscribed in the original triangle having its centre at the orthocentre.

34. Q lies on the common circle of curvature,

hence $$PQ = 4PT.$$

35. AB, BC are equally inclined to the axis, hence since the angles at A and C are equal, AD, DC are equally inclined to the axis.

Hence the tangent at D and AC are equally inclined to the axis.

Therefore the tangents at B and D are parallel.

36. The volume cut off varies as the area VAA' and BB'; and the area VAA' varies as $AV \cdot VA'$ or $AD \cdot A'D'$, that is BC^2.

37. $$SQ : Pg :: St : tg :: SY : SP :: BC : CD :: PF : AC.$$

Hence $$SQ \cdot AC = PF \cdot Pg = AC^2,$$

or $$AC = SQ.$$

Let QL be the ordinate of Q and let MQ meet the major axis in V.

Then $$CV : PN :: CV : CM :: CL : CM - QL :: CL : PN - QL,$$

and $$SP - AC : CN :: SC : AC,$$

or $$AC \cdot SP = AC^2 + CN \cdot SC = BC^2 + CS \cdot SN.$$

Again $$CN - CL : SP - SQ :: SN : SP,$$

and $$SP - SQ : CN :: SC : AC;$$

hence $CN-CL : CN :: SN.SC : SP.AC;$

therefore $CL : CN :: BC^2 : SP.AC,$

or $CL : SP-SQ :: BC^2 : SP.SC.$

Now $SP-SQ : PN-QL :: SP : PN,$

hence $CL : PN-QL :: BC^2 : PN.SC.$

Hence $CV.SC=BC^2,$

or V is a fixed point.

38. If SL, SM, SN be drawn perpendicular to the given tangents, the circle circumscribing LMN is the auxiliary circle of the conic.

39. If S be the centre of the circumscribed circle, H the orthocentre, the centre of the nine-point circle bisects SH: and if PQR be the triangle, the angles SPQ, HPR are equal: hence S and H are the foci.

40. If Pg, $P'g'$ be the normals at P and P' and gL, $g'L'$ be drawn perpendicular to PP',

then $PL=P'L'$, by Ex. 27, Chapter I.;

hence $gG=Gg'.$

Therefore

$$2SG : SP+SP' :: Sg+Sg' : SP+SP' :: SA : AX.$$

41. Reciprocate the following with respect to S: ASB is a diameter of circle meeting a concentric circle in S, the opposite sides of the quadrilateral formed by tangents through A and B to the inner circle are parallel, and the tangents to the outer circle at the points where it meets the tangent at S are respectively parallel to them.

42. If P, Q be two points on a rod and PS, QS are at right angles to the directions of motion of P and Q, then if R be any point on the rod the direction of motion of R is at right angles to SR.

Hence the directions of motion of all points on the rod envelope a parabola of which S is focus and the rod tangent at the vertex.

43. SP and HQ are parallel to CT.

Hence PSp = complement of SPp = complement of CTP.

QHq = complement of HQq = complement of CTQ.

Hence PSp and QHq are together equal to the supplement of PTQ.

44. Let ST, $S'T$ meet CP in p and p', and CD in d and d'.

Since the angle $PTS = d'TD$,

and $TPp = TDd'$,

the angles SpC, $Cd'S'$ are equal, and d, d', p and p' lie on a circle.

45. If the asymptote and directrix meet in D, SDC is a right angle and if DP be the tangent PSD is a right angle.

Therefore SP is parallel to the asymptote.

46. If PTQ, ptq be two consecutive positions and TV, ty be drawn perpendicular to pt, TQ respectively,

$$tV = Pp + pt - PT = PT + TQ + Qq - tq - PT$$
$$= TQ + Qq - tq = Ty.$$

Hence the tangent at T is equally inclined to PT and TQ.

Hence T lies on a confocal ellipse.

47. CP and CQ are at right angles; hence C, P, D, Q lie on a circle.

48. If PV be the chord through the centre, and pp' the parallel focal chord,

$$PV \cdot CP = 2CD^2,$$

and $$pp' \cdot CA = 2CD^2.$$

Hence $$PV : pp' :: CA : CP.$$

49. SP and QH are both parallel to CT;

hence the angle

$$pCq = pCT + TCq = PHQ + QSP.$$

50. The angle SPY = half the supplement of SPS' = half PSP'.

51. QSP, $Q'SP'$ are right angles and the perpendiculars from Q, Q' on SP, SP' are equal to the perpendiculars on PP'.

Hence QP, $Q'P'$ are tangents at Q and Q' to the parabola of which S is focus and PP' directrix.

And the diameter parallel to PP' is tangent at the vertex, since it bisects SH. BB' is a tangent since SCB is a right angle.

52. SE will evidently envelope a conic of which S is focus and the given circle auxiliary circle.

53. Join SP, the bisector of POS bisects SP in V; since the locus of P is a circle, the locus of V is a circle.

Hence VO envelopes a conic of which S is a focus.

54. The angles RSP and QHV are the complements of PST and QHT.

Hence $RSP + QHV$ = supplement of half $PSQ + PHQ$ = half the angle between the tangents at P and Q, by Ex. 23 on the ellipse.

55. TS, ZS are the interior and exterior bisectors of the angle QSR,

and if TQ meet PZ in F, FS is the exterior bisector of the angle QST.

Hence Q, T, R lie on a conic of which S is focus and PZ directrix and ZT is the tangent at T since ZST is a right angle.

56. If OE be the radius of the sphere,

$OE \cdot AC$ = area OAA'

and varies as $OA \cdot OA'$ or $AD \cdot A'D'$ that is as BC^2.

Hence the latera recta of all the sections are the same.

57. Let PQ', QP' meet the ellipse in U and V,

then since $TP^2 = TQ \cdot TQ'$

and $TQ^2 = TP \cdot TP'$,

$TP : TP' :: TQ' : TQ$,

or PQ', $P'Q$ are parallel.

Hence if CF be parallel to $P'Q$ and PQ',

$$P'V.P'Q : P'P^2 :: CF^2 : CD^2,$$

and $$Q'U.Q'P : Q'Q^2 :: CF^2 : CE^2.$$

Now $$P'P^2 : Q'Q^2 :: TP^2 : TQ^2 :: TQ : TQ',$$

and $$CE^2 : CD^2 :: TQ^2 : TP^2 :: TQ : TQ'.$$

Hence $$P'V.P'Q : Q'U.Q'P :: TQ^2 : TQ'^2 :: P'Q^2 : PQ'^2.$$

Therefore $$P'V : P'Q :: Q'U : Q'P.$$

58. If the tangents at P and Q meet in T, CT which bisects PQ is parallel to $P'Q$, P' being the other extremity of the diameter PCP'.

Hence the angle $$PCT = PP'Q = TPQ.$$

59. S, P, S', g lie on a circle,

hence $$Sg : Pg :: S'G : S'P :: SA : AX.$$

60. BC is parallel to the polar of A, hence AD is parallel to the axis.

Also the angles SAC, DAB are equal: and the angles ABD, ASC are likewise equal.

Therefore $$AC : AS :: AD : AB.$$

61. $$RK : QN :: KC : NC$$

and $$PN : RK :: NG : KG.$$

Hence $$BC : AC :: KC.NG : NC.KG,$$

or $$KC : KG :: AC : BC.$$

Therefore $$CR : CQ :: CK : CN :: AC+BC : AC,$$

or $$CR = AC+BC.$$

Hence if NP meet RL in N', $PN = QN$.

Hence KL passes through P.

Also $$PL = QC = AC,$$

and $$KP = QR = BC.$$

62. $$CT.CN = CA^2$$

and $$CT'.PN = BC^2;$$

hence $CT.CT' : CA.CB :: CA.CB : CN.PN$.
Hence the triangle CTT' varies inversely as the triangle PCN.

63. By Ex. 6. Chap. VII.

$$2SP = 3AC,$$

and if CE be drawn parallel to the tangent at P,

$$PE = AC.$$

64. Let CT which bisects PP' in V, meet the ellipse in Q, and let CE be conjugate to CQ.
Then $PV^2 : CV.VT :: CE^2 : CQ^2 :: PV^2 : CQ^2 - CV^2$.
Hence $CQ^2 = CV.VT + CV^2 = CV.CT$,
or TP, TP' are tangents.

65. See Ex. 82 on the hyperbola.

66. If we reciprocate with respect to a focus the theorem that tangents to an ellipse at right angles intersect on a fixed circle, we find that if the sides of a quadrilateral $ABCD$ subtend each a right angle at a fixed point S the sides envelope an ellipse of which S is a focus. If O be the centre,
the angle OAB = complement of half AOB = complement of $SCB = CBS = SAD$.
Hence O is the other focus.

67. If A', B', C', D' be the points of contact and E', F', G' the points of intersection of $A'C'$, $B'D'$; $A'D'$, $B'C'$; $A'B'$, $C'D'$,
then $E'F'G'$ is a self-conjugate triangle.

If $BA'A$, $C'DF$ meet in F the pole of $A'C'$, F will lie on $F'G'$ the polar of E', since E' lies on $A'C'$ the polar of F.
Similarly $AD'D$, $BB'C$ meet in G on $F'G'$ and AC, BD in E'.
Let $A'D'$, $CC'D$ meet in α, and $B'C'$, $AD'D$ in β.
Then if O be intersection of AC', CD', $\alpha\beta$ is the polar of O.

Now since GB', GD' are tangents and the tangent at C' meets them, the range $CC'DF$ is harmonic and therefore the pencil GB', GC', $G\beta$, GF'.

Hence $B'C'\beta F'$ is a harmonic range.

So $A'D'\alpha H$ is a harmonic range.

Hence $\alpha\beta$ passes through G'.

Since G' lies on the polar O, O lies on BD, the polar of G.

Similarly AB', CA' intersect on BD.

68. By Art. 138 the four points in which two rectangular hyperbolas intersect are such that any one of them is the orthocentre of the triangle formed by the other three: hence any conic through the four points is a rectangular hyperbola.

69. If KVt be drawn parallel to the axis to meet, PQ, ST in V and t, and if KL be perpendicular to PQ, $PL=SQ$ and $PV=VQ$, hence $tV=VK$. Therefore ST SP, SK and the axis form a harmonic pencil.

70. Let DE meet PP' in K; and let PD, $P'E$ meet in H.

Then GH is the polar of K, but K lies on DE the polar of F.

Hence F lies on GH, and FG is parallel to the chords bisected by PP'.

71. Let RR' the common tangent be bisected by PQ the common chord in O.

Then $$RO^2=OR'^2=OQ.OP;$$

hence RR', PQ are equally inclined to the axis.

Hence PR is a diameter, and the diameter of curvature $=2PF=2CD$.

Therefore $$CD^2=CD.PF=AC.BC.$$

72. Reciprocate with respect to S the following theorem: S is taken on the outer of two concentric circles; SY, SZ are drawn perpendicular to a pair of parallel tangents to the two circles; YZ is constant.

73. Let the tangents at P and Q meet in T.

Then the angle $P'RQ' = PTQ =$ supplement of PCQ, by Ex. 58.

Hence P', C, Q' and R lie on a circle.

74. TN is half the difference of QM and $Q'M'$ and RR' is half their sum.

Hence $$R'P = \tfrac{1}{2}\, Q'M' = KR.$$

Now $PN : Q'M' :: TK : 2KR :: PM : QM :: QM : 4SP$.

Hence $PN : QM :: Q'M' : 4SP :: PM' : Q'M'$.

Therefore $PN^2 : PM'^2 :: QM^2 : Q'M'^2 :: PM : PM'$,

or $$PN^2 = PM \,.\, PM'.$$

75. Let $RR'V$ be the diameter bisecting PQ.

Then if PQ meet a common tangent pp' in O,

$Op^2 : OP \,.\, OQ :: Sp : SR :: S'p' : S'R' :: Op'^2 : OP \,.\, OQ$,

or $$Op = Op'.$$

76. Let O, C be the centres of the hyperbola and ellipse to $tbCb't'$ the tangent at C, then since $PN \,.\, Ct = Cb^2$ and $CN \,.\, CO = Ca^2$, the area of the ellipse will be a maximum, when $CN \,.\, PN$ is maximum, that is, when $CN \,.\, NO$ is a maximum, or $ON = NC$.

Hence PP' is a tangent to a similar hyperbola.

77. $$RP \,.\, RP' = RN^2 - PN^2 = RN^2 - 4AS \,.\, AN.$$

Now $RN^2 : 4AS \,.\, AA' :: AN^2 : AA'^2$,

or $RN^2 : 4AS \,.\, AN :: AN : AA'$.

Hence $RN^2 - PN^2 : 4AS \,.\, AN :: A'N : AA'$.

Therefore $RP \,.\, RP' : AN \,.\, A'N :: 4AS : AA'$.

78. Let the tangent at P meet the confocal conic in Q.

Draw CEF parallel to PQ meeting the normal at P in F.

Then $$OP \,.\, PF = CD^2 = SP \,.\, PS' = Cd^2,$$

Cd being conjugate to CP.

Hence O is the pole of PQ with respect to the confocal.

79. Let the tangents meet in U, SU meeting the curve in Q, and let the tangent at Q meet RR' in T. Then, V being point of contact of RR',

$$TSR = TSQ + USP - RSP = TSV + USP' - RSV$$
$$= USP' - RST,$$
$$\therefore\ 2TSR = USP' = RSR';$$
$$\therefore \text{ locus of } T \text{ is tangent at } Q.$$

Or by reciprocation of the theorem,

If ABC be a triangle inscribed in a circle, and DE the diameter perpendicular to AC, DB and EB bisect the angle B and its supplement.

80. Reciprocate with respect to any point S the theorem that if two points on a circle be given, the pole of PQ with respect to that circle lies on the line bisecting PQ at right angles.

81. $$PQ \cdot PR = PE \cdot PF = AC \cdot BC$$

and $$QR = CE = AC - BC.$$

Hence $$PQ = BC \text{ and } PR = AC.$$

Also ER is parallel to CQG.

Hence $$PG : CD :: PG : PE :: BC : AC$$

and $$PG \cdot PF = BC^2.$$

Hence CQ, CR are the axes.

82. This is a particular case of Art. 195, since the second point where AE meets the curve is at an infinite distance, hence $AE = EK$.

83. The circle of curvature is greatest at the extremity of the minor axis.

Hence BO the direction of the minor axis is given.

And $BC \cdot BO = AC^2 = SB^2$, O being the centre of curvature.

Hence S lies on the circle of which BO is diameter.

84. $$Ca : Cb :: Ba \cdot Ac : bA \cdot cB$$

and $$Ca : Cb' :: Ba \cdot Ac' : b'A \cdot c'B,$$

by Todhunter's Euclid, Art. 59.

And $Ac.Ac' : Ab.Ab'$ in ratio of squares on parallel diameters.

Hence BC is the tangent at a.

85. This depends on the fact that any chord is bisected by the diameter through the intersection of the tangents at the ends of the chord.

86. Let P, Q be the points of contact of parallel tangents to the conic and circle.

Then by Art. 132, the angles PCA, QCA are equal.

87. Let CL, CL' be the fixed straight lines, S the fixed point.

Then the angle $LSL' = CSL' + CL'S$,

hence the angles $CL'S$, CSL are equal.

Therefore $CL : CS :: CS : CL'$,

or LL' touches the hyperbola of which CL, CL' are asymptotes and S focus.

88. Since the semivertical angles are complementary they touch one another along their common generating line.

Now $EA : AX :: CE : EO :: OE' : CE' :: EA : AS'$.

Hence S' coincides with X, and similarly S with X'.

89. Draw CRV perpendicular to the tangent at P, CE bisecting PP' and PN parallel to CE.

Then $QO.OQ' : PO.OP' :: CD^2 : CR^2 :: PO.PF : CN.CV :: PO : PE :: 2PO : PP'$.

90. A circle can be described with centre T to touch SP, SQ, HP, HQ.

Hence $SN - NH = SM - MH$,

and TM, TN bisect the angles at M, N.

Hence TM, TN touch a confocal conic passing through M and N.

91. If S and H are the given points, the locus of P is the conic in which the given plane through S intersects the

surface generated by the revolution about SH of a conic of which S and H are foci.

If ST be drawn perpendicular to the plane to meet the directrix plane corresponding to S in T, the cone formed by joining H to all points of the locus of P is a right circular cone of which TH is axis.

92. $PG \cdot Pq = BC^2 = PQ^2$;

$\therefore$ PGQ and PgR are similar triangles.

93. If OL be the ordinate of O,

$LG : GN :: OL : PN :: CL : CN$,

or $LC : LG :: CN : NG :: AC : BC^2$.

Hence $CL : CG :: AC^2 : AC^2 + BC^2$.

Therefore $CO : CP' :: CL : CN :: AC^2 - BC^2 : AC^2 + BC^2$.

94. The polygons $V'SPV$ and $Z'HPZ$ are similar and the perpendicular from C on VZ bisects VZ;

hence if VE be taken on VV' such that $VE = ZZ'$,

then $CE = CV' = CZ'$.

Hence $VV' \cdot ZZ' = VV' \cdot VE = VC^2 - V'C^2 = CA^2 - CA'^2$.

95. The centre is the middle point of CP.

96. TSQ and $TS'Q$ are right angles;

$\therefore$ the middle point of TQ is the centre of the circle $TSQS'$ and is equidistant from S and S'.

97. $TQ : TP :: SQ : ST$,

and $T'P : T'Q' :: ST' : SQ'$;

$\therefore TQ \cdot T'P : TP \cdot T'Q' :: SQ \cdot ST' : SQ' \cdot ST$

$:: SQ \cdot PT' : SQ' \cdot PT$.

98. Draw NE perpendicular to NM, and prove that E is a fixed point in the axis.

99. P and Q are equidistant from the plane of the circular section of the cone, which contains the centre of the section.

100. Produce OC to E so that $CE = OC$;

then PE is parallel to CZ and $EPY = CZO = OPY$; that is, the tangent to the curve bisects the angle OPE.

101. If PEQ be one of the tangents, and ERV the chord,

$$EP^2 : ER.EV :: EQ^2 : ER.EV,$$

for each ratio is that of the parallel focal chords.

102. If FE, FG be the tangents, $F(TEQG)$ is harmonic, and EFG is a right angle.

103. Reciprocate with regard to C the theorem, that, if a circle centre C intersect another circle at right angles at the point E, and CPQ be any chord, $CE^2 = CP.CQ$.

104. Reciprocate the conics into two intersecting circles.

105. Reciprocates into the theorem of the existence of the director circle.

106. If PEQ be the chord required, and $P'EQ'$ a consecutive chord, the areas PEP', QEQ' are ultimately equal, and E, which is the centre of curvature, is the middle point of PQ.

PQ is therefore the diameter of curvature and is inclined to the axis at the same angle as the tangent, i.e. half a right angle.

107. The pole F of the straight line is fixed, and P, the point of contact of a tangent, is the foot of the perpendicular from F on the normal.

108. The angle $MNC = LCN = LCN$, if l be the point where the tangent meets the other asymptote.

$\therefore MN$ is parallel to Cl, and passes through P the middle point of Ll.

109. The diameter of curvature being the same for both, it follows that $SP : S'P$ is a constant ratio.

110.

$$CK'^2 = CP^2 + PK'^2 + 2PK'.PF = AC^2 + BC^2 + 2AC.BC;$$
$$\therefore CK' = AC + BC.$$

111. If P, V, R, Q be the points of contact of AB, BC, CD, DA,

$$2ASB = PSV + PSQ,$$

and two right angles

$$= PSV + ASP + VSC = PSV + ASQ + CSR;$$

$$\therefore PSV = QSR = 2QSD,$$

and $$2ASB = 2QSD + PSQ = 2ASD.$$

$$\therefore ASB = ASD = \text{a right angle.}$$

Also $$ASP + DSR = ASD,$$

$\therefore PSQ$ is a straight line and PA, RD intersect on the directrix.

112. If the tangent at P meet the director circle in R and T, perpendiculars to the tangent through R and T are tangents to the ellipse.

Draw PE parallel to CR, meeting CT in V and take $CQ^2 = CV \cdot CT$; similarly find the point D on CR;

then CQ and CD are conjugate diameters, and the construction is completed in Art. 216.

113. $Q'R'$ meets the axis in T, the pole of NP;

$\therefore$ the tangents at Q', R', meet at a point E on NP.

Let CE meet $Q'R'$ in Y;

then $$EN \cdot PN = EN^2 - EN \cdot EP = EC^2 - CN^2 - CY \cdot CE,$$

$$= EC \cdot CY - CN^2 = CA^2 - CN^2 = P'N^2.$$

$$\therefore EN : P'N = P'N : PN = AC : BC = Q'M : QM,$$

and the tangent at Q passes through P'.

114. If S' be the other focus of the fixed ellipse, and H of the moving ellipse,

$$S'P = HP \text{ and } S'Q = HQ.$$

Join $S'H$ meeting the chord in Z, and let fall SY the perpendicular on the chord;

then $SY \cdot S'Z = SY \cdot HZ = BC^2$, and in the chord touches a confocal conic.

115. Let O be the centre of the circle, PQ a chord of intersection not perpendicular to the axis, meeting an asymptote in L, and the axis in K.

$$\text{angle } EOC = 90^0 - LKC = 90^0 - LCA + CLK$$
$$= 45^0 + CLK = LCA + LCE = ECO,$$

$\therefore$ ECO is isosceles, and E lies on a fixed line perpendicular to the axis.

116. Project the ellipse into a circle, and prove that the angle $DQP = QPR$, observing that the tangent and common chord are equally inclined to the axis.

117. Reiprocates into the following:

If S be a fixed point, and SK a tangent to a circle centre C, and if TE be any other tangent from a point T, and the angle $CTE = CSK$, the locus of T is a circle passing through S.

118. SY, HZ being perpendiculars on the tangent,

$$Pq : HZ :: PT : TZ :: TR : TC :: PR : PE;$$
$$\therefore Pq : PR :: HZ : AC :: HP . BC : AC . CD$$
$$:: HP . PG : CD^2 :: PG : S'P;$$

$\therefore$ Rq is parallel to SG.

119. If PT, QT and pt, qt be two near positions and tM, Tm be drawn perpendicular to PT, Qt,

then $tM : Tm$ in the ratio compounded of $PT : QT$ or $CD : CE$ and $Pp . QO' : Qq . PO$ or $PF : QF' : QO', PO$ being the radii of curvature.

Hence $tM = Tm$, or the normal at T to the locus of T bisects PTQ.

Therefore T lies on a confocal ellipse.

120. Referring to the figure of Art. 148, and drawing the lines, a circle can be drawn through $ADOG$;

$$\therefore \text{ angle } DOA = DGA = 90^0 - GVA = AEC,$$

and the triangles AOD, ACE are similar.

$$\therefore AO : AD :: CE : AC,$$

or
$$AO . AC = AD . A'D' = BC^2.$$

121. Referring to the preceding theorem, describe a sphere centre G and radius GA; the tangent from any point of the ellipse to this sphere will be equal to the tangent from P to the circle of curvature.

Describing a similar sphere with centre G', the sum of the tangents $= FA' = SS'$.

122. In the second figure of Art. 144, take T any point in the tangent at P and let C be the centre of the upper sphere.

Then CRP and CSP are right angles, $PR = PS$, and $TR = TS$ these lines being tangents.

$\therefore$ T lies in the plane through CP perpendicular to the plane $CRPS$;

$\therefore$ angle $SPT = RPT$, and $RTP = STP$;

similarly $R'TP = S'TP$.

Hence $RTR' = STP + S'TP = STP + STQ = PTQ$,

TQ being the other tangent.

123. Produce ER to E' making RE' equal to RE.

Then the polar of E' passes through E.

Now C is the pole of PQ which passes through E.

Hence CE' is the polar of E and is therefore parallel to ARB.

Hence CE is bisected by AB.

Again CE bisects in E the polar of E' which is parallel to AB.

Therefore CE bisects AB.

Therefore $ACBE$ is a parallelogram.

124. Let O be the centre of the conic which touches the sides AB, BC, CD, DA in E, F, G, H.

Then since OA, OB, OC, OD bisect HE, EF, FG, GH the sum of the areas of the triangles AOB, COD is half the area of the quadrilateral.

Let AB, CD meet in K and let O, O' be two positions of O.

Draw OM, $O'M'$ perpendicular to AB and ON, $O'N'$ perpendicular to CD.

Also draw $O'K$, $O'L$ perpendicular to OM and ON:

then $OM.AB+ON.CD=O'M'.AB+O'N'.CD$.

Hence $OK.AB=OL\ \ CD$

Therefore $OL : OK$ in a constant ratio.

Hence the locus is a straight line.

125. By Art. 241 the sum or difference of the tangents is proportional to the distance between the ordinates of the points where the circles touch the curve according as the point does or does not lie between those ordinates.

EXAMPLES OMITTED.

CHAPTER III.

60. If P be the corresponding point on the ellipse, SZ the perpendicular,

$$SL : AC :: CT-CS : CT :: AC^2-CS.CN : AC^2$$
$$:: SP : AC;$$
$$\therefore SZ=SP.$$

101. CT bisects PQ and is parallel to SP;
$\therefore$ T is the foot of the perpendicular from S' on PT. See Cor. (3), Art. (66).

102. $SC : CY$ is a given ratio and SY is fixed.

103. Take p a point near and let the focal chord $p'Sq'$ meet pq in O;

$$PQ : p'q' :: pO.Oq : p'O.Oq' :: ST.Oq : Sp'.Oq'$$
$$:: ST.pq : Sp.p'q' \text{ ultimately.}$$

104. Dropping perpendiculars from the focus on the sides, their feet are the middle points, and, as they lie on a circle, form a rectangle; the diagonals, intersecting in H, are therefore at right angles, and SAD can be proved equal to HAB.

105. If CD, CP meet the directrix in E and G, ES is perpendicular to the chord of contact of tangents from E, which is parallel to CP.

106. If CP, DC meet the tangent at A in Q and R,

prove that $AQ \cdot AR = BC^2 = AS \cdot AS'$.

Then $QSA = ARS'$, and $QRA = AQS'$.

CHAPTER IV.

101. The chord QR is inclined to the axis at the same angle as the tangent at P and is therefore always parallel to a fixed line.

102. TP and the asymptote subtend equal angles at S';

$$\therefore PS'T = S'TC = STP.$$

103.
$$\begin{aligned} SF^2 &= FX^2 + SX^2 = CF^2 - CX^2 + SX^2 \\ &= CF^2 + CS^2 - 2CS \cdot CX \\ &= CF^2 + CS^2 - 2CA^2 = CF^2 - CA^2 + CB^2 \\ &= \text{square of tangent from } F \\ &= FA \cdot FB. \end{aligned}$$

CHAPTER V.

40. If the tangent at Q meet in V the line joining the fixed points A and B,

$$VQ^2 = VA \cdot VB.$$

41. If the chord QR meet the tangent at P in E,

$$RPL = QPE = PRQ.$$

42. If D be the point,

$$CD \,.\, CT = 2AC^2 \text{ and } CD = AC.$$

43. $CP = CD$ and are equally inclined to the asymptote.

44. Let BAD be the given difference, and draw CL parallel to AD, meeting BA in L. CAL, CBL are similar triangles;

$$CL^2 = AL \,.\, BL.$$

45. If tQT, $t'QT'$ be the tangents, and SM perpendicular to the axis,

$$SQM = SQT - MQT = S'Qt' - CtT$$
$$= QS'M + QT'C - CtT = QS'M;$$
$$\therefore QM^2 = SM \,.\, S'M.$$

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AT THE UNIVERSITY PRESS.

May 1881.

A CLASSIFIED LIST

OF

EDUCATIONAL WORKS

PUBLISHED BY

GEORGE BELL & SONS.

Full Catalogues will be sent post free on application.

BIBLIOTHECA CLASSICA.

A Series of Greek and Latin Authors, with English Notes, edited by eminent Scholars. 8*vo.*

Æschylus. By F. A. Paley, M.A. 18*s.*

Cicero's Orations. By G. Long, M.A. 4 vols. 16*s.*, 14*s.*, 16*s.*, 18*s.*

Demosthenes. By R. Whiston, M.A. 2 vols. 16*s.* each.

Euripides. By F. A. Paley, M.A. 3 vols. 16*s.* each.

Homer. By F. A. Paley, M.A. Vol. I. 12*s.*; Vol. II. 14*s.*

Herodotus. By Rev. J. W. Blakesley, B.D. 2 vols. 32*s.*

Hesiod. By F. A. Paley, M.A. 10*s.* 6*d.*

Horace. By Rev. A. J. Macleane, M.A. 18*s.*

Juvenal and Persius. By Rev. A. J. Macleane, M.A. 12*s.*

Plato. By W. H. Thompson, D.D. 2 vols. 7*s.* 6*d.* each.

Sophocles. Vol. I. By Rev. F. H. Blaydes, M.A. 18*s.*

——— Vol. II. Philoctetes. Electra. Ajax and Trachiniæ. By F. A. Paley, M.A. 12*s.*

Tacitus: The Annals. By the Rev. P. Frost. 15*s.*

Terence. By E. St. J. Parry, M.A. 18*s.*

Virgil. By J. Conington, M.A. 3 vols. 14*s.* each.

An Atlas of Classical Geography; Twenty-four Maps. By W. Hughes and George Long, M.A. New edition, with coloured outlines. Imperial 8vo. 12*s.* 6*d.*

Uniform with above.

A Complete Latin Grammar. By J. W. Donaldson, D.D. 3rd Edition. 14*s.*

GRAMMAR-SCHOOL CLASSICS.

A Series of Greek and Latin Authors, with English Notes. Fcap. 8*vo.*

Cæsar: De Bello Gallico. By George Long, M.A. 5*s.* 6*d.*

——— Books I.–III. For Junior Classes. By G. Long, M.A. 2*s.* 6*d.*

Catullus, Tibullus, and Propertius. Selected Poems. With Life. By Rev. A. H. Wratislaw. 3*s.* 6*d.*

Cicero: De Senectute, De Amicitia, and Select Epistles. By George Long, M.A. 4*s.* 6*d.*

Cornelius Nepos. By Rev. J. F. Macmichael. 2*s.* 6*d.*

Homer: Iliad. Books I.–XII. By F. A. Paley, M.A. 6*s.* 6*d.*

Horace. With Life. By A. J. Macleane, M.A. 6*s.* 6*d.* [In 2 parts. 3*s.* 6*d.* each.]

Juvenal: Sixteen Satires. By H. Prior, M.A. 4*s.* 6*d.*

Martial: Select Epigrams. With Life. By F. A. Paley, M.A. 6*s.* 6*d.*

Ovid: the Fasti. By F. A. Paley, M.A. 5*s.*

Sallust: Catilina and Jugurtha. With Life. By G. Long, M.A. 5*s.*

Tacitus: Germania and Agricola. By Rev. P. Frost. 3*s.* 6*d.*

Virgil: Bucolics, Georgics, and Æneid, Books I.–IV. Abridged from Professor Conington's Edition. 5*s.* 6*d.*
(The Bucolics and Georgics in one volume. 3*s.*)

——— Æneid, Books V.–XII. Abridged from Professor Conington's Edition. 5*s.* 6*d.*

Xenophon: The Anabasis. With Life. By Rev. J. F. Macmichael. 5*s.*

——— The Cyropædia. By G. M. Gorham, M.A. 6*s.*

——— Memorabilia. By Percival Frost, M.A. 4*s.* 6*d.*

A Grammar-School Atlas of Classical Geography, containing Ten selected Maps. Imperial 8vo. 5*s.*

Uniform with the Series.

The New Testament, in Greek. With English Notes, &c. By Rev. J. F. Macmichael. 7*s.* 6*d.*

CAMBRIDGE GREEK AND LATIN TEXTS.

Æschylus. By F. A. Paley, M.A. 3*s.*

Cæsar: De Bello Gallico. By G. Long, M.A. 2*s.*

Cicero: De Senectute et de Amicitia, et Epistolæ Selectæ. By G. Long, M.A. 1*s.* 6*d.*

Ciceronis Orationes. Vol. I. (in Verrem.) By G. Long, M.A. 3*s.* 6*d.*

Euripides. By F. A. Paley, M.A. 3 vols. 3*s.* 6*d.* each.

Herodotus. By J. G. Blakesley, B.D. 2 vols. 7*s.*

Homeri Ilias. I.–XII. By F. A. Paley, M.A. 2*s.* 6*d.*

Horatius. By A. J. Macleane, M.A. 2*s.* 6*d.*

Juvenal et Persius. By A. J. Macleane, M.A. 1*s.* 6*d.*

Lucretius. By H. A. J. Munro, M.A. 2*s.* 6*d.*

Sallusti Crispi Catilina et Jugurtha. By G. Long, M.A. 1*s.* 6*d.*

Sophocles. By F. A. Paley, M.A. [*In the press.*

Terenti Comœdiæ. By W. Wagner, Ph.D. 3*s.*

Thucydides. By J. G. Donaldson, D.D. 2 vols. 7*s.*

Virgilius. By J. Conington, M.A. 3*s.* 6*d.*

Xenophontis Expeditio Cyri. By J. F. Macmichael, B.A. 2*s.* 6*d.*

Novum Testamentum Græcum. By F. H. Scrivener, M.A. 4*s.* 6*d.* *An edition* with wide margin for notes, half bound, 12*s.*

CAMBRIDGE TEXTS WITH NOTES.

A Selection of the most usually read of the Greek and Latin Authors, Annotated for Schools. Fcap. 8vo. 1s. 6d. each., with exceptions.

Euripides. Alcestis.—Medea.—Hippolytus.—Hecuba.—Bacchæ. Ion. 2s.—Orestes.—Phoenissæ. By F. A. Paley, M.A.

Æschylus. Prometheus Vinctus.—Septem contra Thebas.—Agamemnon.—Persæ.—Eumenides. By F. A. Paley, M.A. 1s.

Sophocles. Œdipus Tyrannus. By F. A. Paley, M.A. [*In the press.*

Homer. Iliad. Book I. By F. A. Paley, M.A. 1s.

Cicero's De Senectute—De Amicitia and Epistolæ Selectæ. By G. Long, M.A.

Ovid. Selections. By A. J. Macleane, M.A.

Others in preparation.

PUBLIC SCHOOL SERIES.

A Series of Classical Texts, annotated by well-known Scholars. Cr. 8vo.

Aristophanes. The Peace. By F. A. Paley, M.A. 4s. 6d.

——— The Acharnians. By F. A. Paley, M.A. 4s. 6d.

——— The Frogs. By F. A. Paley, M.A. 4s. 6d.

Cicero. The Letters to Atticus. Bk. I. By A. Pretor, M.A. 4s. 6d.

Demosthenes de Falsa Legatione. By R. Shilleto, M.A. 6s.

——— The Law of Leptines. By B. W. Beatson, M.A. 3s. 6d.

Plato. The Apology of Socrates and Crito. By W. Wagner, Ph.D. 4th Edition. 4s. 6d.

——— The Phædo. By W. Wagner, Ph.D. 5s. 6d.

——— The Protagoras. By W. Wayte, M.A. 4s. 6d.

——— The Euthyphro. 2nd edition. By G. H. Wells. 3s.

——— The Euthydemus. By G. H. Wells. [*In the press.*

Plautus. The Aulularia. By W. Wagner, Ph.D. 2nd edition. 4s. 6d.

——— Trinummus. By W. Wagner, Ph.D. 2nd edition. 4s. 6d.

——— The Menaechmei. By W. Wagner, Ph.D. 4s. 6d.

Sophoclis Trachiniæ. By A. Pretor, M.A. 4s. 6d.

Terence. By W. Wagner, Ph.D. 10s. 6d.

Theocritus. By F. A. Paley, M.A. 4s. 6d.

Others in preparation.

CRITICAL AND ANNOTATED EDITIONS.

Ætna. By H. A. J. Munro, M.A. 3s. 6d.

Aristophanis Comœdiæ. By H. A. Holden, LL.D. 8vo. 2 vols. 23s. 6d. Plays sold separately.

——— Pax. By F. A. Paley, M.A. Fcap. 8vo. 4s. 6d.

Catullus. By H. A. J. Munro, M.A. 7s. 6d.

Corpus Poetarum Catinorum. Edited by Walker. 1 vol. 8vo. 18s.

Horace. Quinti Horatii Flacci Opera. By H. A. J. Munro, M.A. Large 8vo. 1l. 1s.

Livy. The first five Books. By J. Prendeville. 12mo. roan, 5s. Or Books I.-III. 3s. 6d. IV. and V. 3s. 6d.

Lucretius. Titi Lucretii Cari de Rerum Natura Libri Sex. With a Translation and Notes. By H. A. J. Munro, M.A. 2 vols. 8vo. Vol. I. Text. Vol. II. Translation. (Sold separately.)

Ovid. P. Ovidii Nasonis Heroides XIV. By A. Palmer, M.A. 8vo. 6*s*.

Propertius. Sex Aurelii Propertii Carmina. By F. A. Paley, M.A. 8vo. Cloth, 9*s*.

Sex. Propertii Elegiarum. Lib. IV. By A. Palmer. Fcap. 8vo. 5*s*.

Sophocles. The Ajax. By C. E. Palmer, M.A. 4*s*. 6*d*.

Thucydides. The History of the Peloponnesian War. By Richard Shilleto, M.A. Book I. 8vo. 6*s*. 6*d*. (Book II. *in the press*.)

LATIN AND GREEK CLASS-BOOKS.

Auxilia Latina. A Series of Progressive Latin Exercises. By M. J. B. Baddeley, M.A. Fcap. 8vo. Part I. Accidence. 1*s*. 6*d*. Part II. 2nd Edition, 2*s*. Key, 2*s*. 6*d*.

Latin Prose Lessons. By Prof. Church, M.A. 3rd Edit. Fcap. 8vo. 2*s*. 6*d*.

Latin Exercises and Grammar Papers. By T. Collins, M.A. 2nd Edition. Fcap. 8vo. 2*s*. 6*d*.

Unseen Papers in Prose and Verse. With Examination Questions. By T. Collins, M.A. 2nd edition. Fcap. 8vo. 2*s*. 6*d*.

Analytical Latin Exercises. By C. P. Mason, B.A. 2nd Edit. 3*s*. 6*d*.

Scala Græca: a Series of Elementary Greek Exercises. By Rev. J. W. Davis, M.A., and R. W. Baddeley, M.A. 3rd Edition. Fcap. 8vo. 2*s*. 6*d*.

Greek Verse Composition. By G. Preston, M.A. Crown 8vo. 4*s*. 6*d*.

BY THE REV. P. FROST, M.A., ST. JOHN'S COLLEGE, CAMBRIDGE.

Eclogæ Latinæ; or, First Latin Reading-Book, with English Notes and a Dictionary. New Edition. Fcap. 8vo. 2*s*. 6*d*.

Materials for Latin Prose Composition. New Edition. Fcap. 8vo. 2*s*. 6*d*. Key, 4*s*.

A Latin Verse-Book. An Introductory Work on Hexameters and Pentameters. New Edition. Fcap. 8vo. 3*s*. Key, 5*s*.

Analecta Græca Minora, with Introductory Sentences, English Notes, and a Dictionary. New Edition. Fcap. 8vo. 3*s*. 6*d*.

Materials for Greek Prose Composition. New Edit. Fcap. 8vo. 3*s*. 6*d*. Key, 5*s*.

Florilegium Poeticum. Elegiac Extracts from Ovid and Tibullus. New Edition. With Notes. Fcap. 8vo. 3*s*.

BY THE REV. F. E. GRETTON.

A First Cheque-book for Latin Verse-makers. 1*s*. 6*d*.

A Latin Version for Masters. 2*s*. 6*d*.

Reddenda; or Passages with Parallel Hints for Translation into Latin Prose and Verse. Crown 8vo. 4*s*. 6*d*.

Reddenda Reddita (*see next page*).

BY H. A. HOLDEN, LL.D.

Foliorum Silvula. Part I. Passages for Translation into Latin Elegiac and Heroic Verse. 9th Edition. Post 8vo. 7*s*. 6*d*.

——— Part II. Select Passages for Translation into Latin Lyric and Comic Iambic Verse. 3rd Edition. Post 8vo. 5*s*.

——— *Part III.* Select Passages for Translation into Greek Verse. *3rd Edition.* Post 8vo. 8*s*.

Folia Silvulæ, sive Eclogæ Poetarum Anglicorum in Latinum et Græcum conversæ. 8vo. Vol. I. 10*s.* 6*d.* Vol. II. 12*s.*

Foliorum Centuriæ. Select Passages for Translation into Latin and Greek Prose. 7th Edition. Post 8vo. 8*s.*

TRANSLATIONS, SELECTIONS, &c.

*** Many of the following books are well adapted for School Prizes.

Æschylus. Translated into English Prose by F. A. Paley, M.A. 2nd Edition. 8vo. 7*s.* 6*d.*

——— Translated into English Verse by Anna Swanwick. Post 8vo. [*In the press.*

——— Folio Edition, with 33 Illustrations after Flaxman. 2*l.* 2*s.*

Anthologia Græca. A Selection of Choice Greek Poetry, with Notes. By F. St. John Thackeray. *4th and Cheaper Edition.* 16mo. 4*s.* 6*d.*

Anthologia Latina. A Selection of Choice Latin Poetry, from Nævius to Boëthius, with Notes. By Rev. F. St. John Thackeray. Revised and Cheaper Edition. 16mo. 4*s.* 6*d.*

Horace. The Odes and Carmen Sæculare. In English Verse by J. Conington, M.A. 8th edition. Fcap. 8vo. 5*s.* 6*d.*

——— The Satires and Epistles. In English Verse by J. Conington, M.A. 5th edition. 6*s.* 6*d.*

——— Illustrated from Antique Gems by C. W. King, M.A. The text revised with Introduction by H. A. J. Munro, M.A. Large 8vo. 1*l.* 1*s.*

Horace's Odes. Englished and Imitated by various hands. Edited by C. W. F. Cooper. Crown 8vo. 6*s.* 6*d.*

Mvsæ Etonenses, sive Carminvm Etonæ Conditorvm Delectvs. By Richard Okes. 2 vols. 8vo. 15*s.*

Propertius. Verse translations from Book V., with revised Latin Text. By F. A. Paley, M.A. Fcap. 8vo. 3*s.*

Plato. Gorgias. Translated by E. M. Cope, M.A. 8vo. 7*s.*

——— Philebus. Translated by F. A. Paley, M.A. Small 8vo. 4*s.*

——— Theætetus. Translated by F. A. Paley, M.A. Small 8vo, 4*s.*

——— Analysis and Index of the Dialogues. By Dr. Day. Post 8vo. 5*s.*

Reddenda Reddita: Passages from English Poetry, with a Latin Verse Translation. By F. E. Gretton. Crown 8vo. 6*s.*

Sabrinæ Corolla in hortulis Regiæ Scholæ Salopiensis contexuerunt tres viri floribus legendis. Editio tertia. 8vo. 8*s.* 6*d.*

Sertum Carthusianum Floribus trium Seculorum Contextum. By W. H. Brown. 8vo. 14*s.*

Theocritus. In English Verse, by C. S. Calverley, M.A. Crown 8vo. 7*s.* 6*d.*

Translations into English and Latin. By C. S. Calverley, M.A. Post 8vo. 7*s.* 6*d.*

——— By R. C. Jebb, M.A.; H. Jackson, M.A., and W. E. Currey, M.A. Crown 8vo. 8*s.*

——— into Greek and Latin Verse. By R. C. Jebb. 4to. cloth gilt. 10*s.* 6*d.*

Between Whiles. Translations by B. H. Kennedy. Crown 8vo. 6*s.*

The Enunciations and Figures to Euclid's Elements. By Rev. J. Brasse, D.D. 3rd Edition. Fcap. 8vo. 1s. On Cards, in case, 5s. 6d. Without the Figures, 6d.

Exercises on Euclid and in Modern Geometry. By J. McDowell, B.A. Crown 8vo. 2nd Edition revised. 6s.

Geometrical Conic Sections. By W. H. Besant, M.A. 3rd Edit. 4s. 6d.

Elementary Geometrical Conic Sections. By W. H. Besant, M.A. [*In the Press.*

The Geometry of Conics. By C. Taylor, M.A. 2nd Edit. 8vo. 4s. 6d.

Solutions of Geometrical Problems, proposed at St. John's College from 1830 to 1846. By T. Gaskin, M.A. 8vo. 12s.

TRIGONOMETRY.

The Shrewsbury Trigonometry. By J. C. P. Aldous. Crown 8vo. 2s.

Elementary Trigonometry. By T. P. Hudson, M.A. 3s. 6d.

Elements of Plane and Spherical Trigonometry. By J. Hind, M.A. 5th Edition. 12mo. 6s.

An Elementary Treatise on Mensuration. By B. T. Moore, M.A. 5s.

ANALYTICAL GEOMETRY AND DIFFERENTIAL CALCULUS.

An Introduction to Analytical Plane Geometry. By W. P. Turnbull, M.A. 8vo. 12s.

Problems on the Principles of Plane Co-ordinate Geometry. By W. Walton, M.A. 8vo. 16s.

Trilinear Co-ordinates, and Modern Analytical Geometry of Two Dimensions. By W. A. Whitworth, M.A. 8vo. 16s.

An Elementary Treatise on Solid Geometry. By W. S. Aldis, M.A. 2nd Edition revised. 8vo. 8s.

Geometrical Illustrations of the Differential Calculus. By M. B. Pell. 8vo. 2s. 6d.

Elementary Treatise on the Differential Calculus. By M. O'Brien, M.A. 8vo. 10s. 6d.

Elliptic Functions, Elementary Treatise on. By A. Cayley, M.A. Demy 8vo. 15s.

MECHANICS & NATURAL PHILOSOPHY.

Statics, Elementary. By H. Goodwin, D.D. Fcap. 8vo. 2nd Edition. 3s.

Dynamics, A Treatise on Elementary. By W. Garnett, M.A. *2nd Edition.* Crown 8vo. 6s.

Statics and Dynamics, Problems in. By W. Walton, M.A. New Edition. Crown 8vo. 6s.

Theoretical Mechanics, Problems in. By W. Walton. 2nd Edit. revised and enlarged. Demy 8vo. 16s.

Hydrostatics. By W. H. Besant, M.A. Fcap. 8vo. 9th Edition. 4s.

Hydromechanics, A Treatise on. By W. H. Besant, M.A. 8vo. New Edition revised. 10s. 6d.

Dynamics of a Particle, A Treatise on the. By W. H. Besant, M.A. [*Preparing.*

Dynamics of a Rigid Body, Solutions of Examples on the. By W. N. Griffin, M.A. 8vo. 6s. 6d.

Motion, An Elementary Treatise on. By J. R. Lunn, M.A. 7s. 6d.

Optics, Geometrical. By W. S. Aldis, M.A. Fcap. 8vo. 3s. 6d.

Double Refraction, A Chapter on Fresnel's Theory of. By W. S. Aldis, M.A. 8vo. 2s.

Optics, An Elementary Treatise on. By Prof. Potter. Part I. 3rd Edition. 9s. 6d. Part II. 12s. 6d.

Optics, Physical; or the Nature and Properties of Light. By Prof. Potter, A.M. 6s. 6d. Part II. 7s. 6d.

Heat, An Elementary Treatise on. By W. Garnett, M.A. Crown 8vo. 2nd Edition revised. 3s. 6d.

Geometrical Optics, Figures Illustrative of. From Schelbach. By W. B. Hopkins. Folio. Plates. 10s. 6d.

Newton's Principia, The First Three Sections of, with an Appendix; and the Ninth and Eleventh Sections. By J. H. Evans, M.A. 5th Edition. Edited by P. T. Main, M.A. 4s.

Astronomy, An Introduction to Plane. By P. T. Main, M.A. Fcap. 8vo. cloth. 4s.

Astronomy, Practical and Spherical. By R. Main, M.A. 8vo. 14s.

Astronomy, Elementary Chapters on, from the 'Astronomie Physique' of Biot. By H. Goodwin, D.D. 8vo. 3s. 6d.

Pure Mathematics and Natural Philosophy, A Compendium of Facts and Formulæ in. By G. R. Smalley. Fcap. 8vo. 3s. 6d.

Elementary Course of Mathematics. By H. Goodwin, D.D. 6th Edition. 8vo. 16s.

Problems and Examples, adapted to the 'Elementary Course of Mathematics.' 3rd Edition. 8vo. 5s.

Solutions of Goodwin's Collection of Problems and Examples. By W. W. Hutt, M.A. 3rd Edition, revised and enlarged. 8vo. 9s.

Pure Mathematics, Elementary Examples in. By J. Taylor. 8vo. 7s. 6d.

Euclid, Mechanical. By the late W. Whewell, D.D. 5th Edition. 5s.

Mechanics of Construction. With numerous Examples. By S. Fenwick, F.R.A.S. 8vo. 12s.

Anti-Logarithms, Table of. By H. E. Filipowski. 3rd Edition. 8vo. 15s.

Pure and Applied Calculation, Notes on the Principles of. By Rev. J. Challis, M.A. Demy 8vo. 15s.

Physics, The Mathematical Principle of. By Rev. J. Challis, M.A. Demy 8vo. 5s.

HISTORY, TOPOGRAPHY, &c.

Rome and the Campagna. By R. Burn, M.A. With 85 Engravings and 26 Maps and Plans. With Appendix. 4to. 3l. 3s.

Old Rome. A Handbook for Travellers. By R. Burn, M.A. With Maps and Plans. Demy 8vo. 10s. 6d.

Modern Europe. By Dr. T. H. Dyer. 2nd Edition, revised and continued. 5 vols. Demy 8vo. 2l. 12s. 6d.

The History of the Kings of Rome. By Dr. T. H. Dyer. 8vo. 16s.

A Plea for Livy. By Dr. T. H. Dyer. 8vo. 1s.

Roma Regalis. By Dr. T. H. Dyer. 8vo. 2s. 6d.

The History of Pompeii: its Buildings and Antiquities. By T. H. Dyer. 3rd Edition, brought down to 1874. Post 8vo. 7s. 6d.

Ancient Athens: its History, Topography, and Remains. By T. H. Dyer. Super-royal 8vo. Cloth. 1l. 5s.

The Decline of the Roman Republic. By G. Long. 5 vols. 8vo. 14s. each.

A History of England during the Early and Middle Ages. By C. H. Pearson, M.A. 2nd Edition revised and enlarged. 8vo. Vol. I. 16s. Vol. II. 14s.

Historical Maps of England. By C. H. Pearson. Folio. 2nd Edition revised. 31s. 6d.

History of England, 1800–15. By Harriet Martineau, with new and copious Index. 1 vol. 3s. 6d.

History of the Thirty Years' Peace, 1815–46. By Harriet Martineau. 4 vols. 3s. 6d. each.

A Practical Synopsis of English History. By A. Bowes. 4th Edition. 8vo. 2s.

Student's Text-Book of English and General History. By D. Beale. Crown 8vo. 2s. 6d.

Lives of the Queens of England. By A. Strickland. Library Edition, 8 vols. 7s. 6d. each. Cheaper Edition, 6 vols. 5s. each. Abridged Edition, 1 vol. 6s. 6d.

Eginhard's Life of Karl the Great (Charlemagne). Translated with Notes, by W. Glaister, M.A., B.C.L. Crown 8vo. 4s. 6d.

Outlines of Indian History. By A. W. Hughes. Small post 8vo. 3s. 6d.

The Elements of General History. By Prof. Tytler. New Edition, brought down to 1874. Small post 8vo. 3s. 6d.

ATLASES.

An Atlas of Classical Geography. 24 Maps. By W. Hughes and G. Long, M.A. New Edition. Imperial 8vo. 12s. 6d.

A Grammar-School Atlas of Classical Geography. Ten Maps selected from the above. New Edition. Imperial 8vo. 5s.

First Classical Maps. By the Rev. J. Tate, M.A. 3rd Edition. Imperial 8vo. 7s. 6d.

***Standard Library* Atlas of Classical Geography.** Imp. 8vo. 7s. 6d.

PHILOLOGY.

WEBSTER'S DICTIONARY OF THE ENGLISH LANGUAGE. With Dr. Mahn's Etymology. 1 vol., 1628 Pages, 3000 Illustrations. 21*s.* With Appendices and 70 additional pages of Illustrations, 1919 Pages, 31*s.* 6*d.*

'THE BEST PRACTICAL ENGLISH DICTIONARY EXTANT.'—*Quarterly Review*, 1873.

Prospectuses, with specimen pages, post free on application.

New Dictionary of the English Language. Combining Explanation with Etymology, and copiously illustrated by Quotations from the best Authorities. By Dr. Richardson. New Edition, with a Supplement. 2 vols. 4to. 4*l.* 14*s.* 6*d.*; half russia, 5*l.* 15*s.* 6*d.*; russia, 6*l.* 12*s.* Supplement separately. 4to. 12*s.*

An 8vo. Edit. without the Quotations, 15*s.*; half russia, 20*s.*; russia, 24*s.*

Supplementary English Glossary. By T. L. O. Davies. Demy 8vo. [*In the press.*

Dictionary of Corrupted Words. By Rev. A. S. Palmer. [*In the press.*

The Elements of the English Language. By E. Adams, Ph.D. 15th Edition. Post 8vo. 4*s.* 6*d.*

Philological Essays. By T. H. Key, M.A., F.R.S. 8vo. 10*s.* 6*d.*

Language, its Origin and Development. By T. H. Key, M.A., F.R.S. 8vo. 14*s.*

Synonyms and Antonyms of the English Language. By Archdeacon Smith. 2nd Edition. Post 8vo. 5*s.*

Synonyms Discriminated. By Archdeacon Smith. Demy 8vo. 16*s.*

Bible English. By T. L. O. Davies. 5*s.*

The Queen's English. A Manual of Idiomand Usage. By the late Dean Alford. 5th Edition. Fcap. 8vo. 5*s.*

Etymological Glossary of nearly 2500 English Words derived from the Greek. By the Rev. E. J. Boyce. Fcap. 8vo. 3*s.* 6*d.*

A Syriac Grammar. By G. Phillips, D.D. 3rd Edition, enlarged. 8vo. 7*s.* 6*d.*

A Grammar of the Arabic Language. By Rev. W. J. Beaumont, M.A. 12mo. 7*s.*

Who Wrote It? A Dictionary of Common Poetical Quotations. 3rd Edition. Fcap. 8vo. 2*s.* 6*d.*

DIVINITY, MORAL PHILOSOPHY, &c.

Novum Testamentum Græcum, Textus Stephanici, 1550. By F. H. Scrivener, A.M., LL.D. New Edition. 16mo. 4*s.* 6*d.* Also on Writing Paper, with Wide Margin. Half-bound. 12*s.*

By the same Author.

Codex Bezæ Cantabrigiensis. 4to. 26*s.*

A Full Collation of the Codex Sinaiticus with the Received Text of the New Testament, with Critical Introduction. 2nd Edition, revised. Fcap. 8vo. 5*s.*

A Plain Introduction to the Criticism of the New Testament. With Forty Facsimiles from Ancient Manuscripts. 2nd Edition. 8vo. 16*s.*

Six Lectures on the Text of the New Testament. For English Readers. Crown 8vo. 6*s.*

The New Testament for English Readers. By the late H. Alford, D.D. Vol. I. Part I. 3rd Edit. 12s. Vol. I. Part II. 2nd Edit. 10s. 6d. Vol. II. Part I. 2nd Edit. 16s. Vol. II. Part II. 2nd Edit. 16s.

The Greek Testament. By the late H. Alford, D.D. Vol. I. 6th Edit. 1l. 8s. Vol. II. 6th Edit. 1l. 4s. Vol. III. 5th Edit. 18s. Vol. IV. Part I. 4th Edit. 18s. Vol. IV. Part II. 4th Edit. 14s. Vol. IV. 1l. 12s.

Companion to the Greek Testament. By A. C. Barrett, M.A. 3rd Edition. Fcap. 8vo. 5s.

Liber Apologeticus. The Apology of Tertullian, with English Notes, by H. A. Woodham, LL.D. 2nd Edition. 8vo. 8s. 6d.

The Book of Psalms. A New Translation, with Introductions, &c. By the Very Rev. J. J. Stewart Perowne, D.D. 8vo. Vol. I. 4th Edition, 18s. Vol. II. 4th Edit. 16s.

——— Abridged for Schools. 3rd Edition. Crown 8vo. **10s. 6d.**

History of the Articles of Religion. By C. H. Hardwick. 3rd Edition. Post 8vo. 5s.

History of the Creeds. By J. R. Lumby, D.D. 2nd Edition. Crown 8vo. 7s. 6d.

Pearson on the Creed. Carefully printed from an early edition. With Analysis and Index by E. Walford, M.A. Post 8vo. 5s.

Doctrinal System of St. John as Evidence of the Date of his Gospel. By Rev. J. J. Lias, M.A. Crown 8vo. 6s.

An Historical and Explanatory Treatise on the Book of Common Prayer. By Rev. W. G. Humphry, B.D. 5th Edition, enlarged. Small post 8vo. 4s. 6d.

The New Table of Lessons Explained. By Rev. W. G. Humphry, B.D. Fcap. 1s. 6d.

A Commentary on the Gospels for the Sundays and other Holy Days of the Christian Year. By Rev. W. Denton, A.M. New Edition. 3 vols. 8vo. 54s. Sold separately.

Commentary on the Epistles for the Sundays and other Holy Days of the Christian Year. By Rev. W. Denton, A.M. 2 vols. 36s. Sold separately.

Commentary on the Acts. By Rev. W. Denton, A.M. Vol. I. 8vo. 18s. Vol. II. 14s.

Notes on the Catechism. By Rev. A. Barry, D.D. 5th Edit. Fcap. 2s.

Catechetical Hints and Helps. By Rev. E. J. Boyce, M.A. 3rd Edition, revised. Fcap. 2s. 6d.

Examination Papers on Religious Instruction. By Rev. E. J. Boyce. Sewed. 1s. 6d.

Church Teaching for the Church's Children. An Exposition of the Catechism. By the Rev. F. W. Harper. Sq. fcap. 2s.

The Winton Church Catechist. Questions and Answers on the Teaching of the Church Catechism. By the late Rev. J. S. B. Monsell, LL.D. 3rd Edition. Cloth, 3s.; or in Four Parts, sewed.

The Church Teacher's Manual of Christian Instruction. By Rev. M. F. Sadler. 21st Thousand. 2s. 6d.

Short Explanation of the Epistles and Gospels of the Christian Year, with Questions. Royal 32mo. 2s. 6d.; calf, 4s. 6d.

Butler's Analogy of Religion; with Introduction and Index by Rev. Dr. Steere. New Edition. Fcap. 3s. 6d.

——— *Three* **Sermons on Human Nature, and Dissertation on** *Virtue.* By W. Whewell, D.D. 4th Edition. Fcap. 8vo. 2s. 6d.

Lectures on the History of Moral Philosophy in England. By W. Whewell, D.D. Crown 8vo. 8*s.*

Kent's Commentary on International Law. By J. T. Abdy, LL.D. New and Cheap Edition. Crown 8vo. 10*s.* 6*d.*

A Manual of the Roman Civil Law. By G. Leapingwell, LL.D. 8vo. 12*s.*

FOREIGN CLASSICS.

A series for use in Schools, with English Notes, grammatical and explanatory, and renderings of difficult idiomatic expressions. Fcap. 8vo.

Schiller's Wallenstein. By Dr. A. Buchheim. New Edit. 6*s.* 6*d.* Or the Lager and Piccolomini, 3*s.* 6*d.* Wallenstein's Tod, 3*s.* 6*d.*

——— **Maid of Orleans.** By Dr. W. Wagner. 3*s.* 6*d.*

——— **Maria Stuart.** By V. Kastner. 3*s.*

Goethe's Hermann and Dorothea. By E. Bell, M.A., and E. Wölfel. 2*s.* 6*d.*

German Ballads, from Uhland, Goethe, and Schiller. By C. L. Bielefeld. 3*s.* 6*d.*

Charles XII., par Voltaire. By L. Direy. 3rd Edition. 3*s.* 6*d.*

Aventures de Télémaque, par Fénélon. By C. J. Delille. 2nd Edition. 4*s.* 6*d.*

Select Fables of La Fontaine. By F. E. A. Gasc. New Edition. 3*s.*

Picciola, by X. B. Saintine. By Dr. Dubuc. 4th Edition. 3*s.* 6*d.*

FRENCH CLASS-BOOKS.

Twenty Lessons in French. With Vocabulary, giving the Pronunciation. By W. Brebner. Post 8vo. 4*s.*

French Grammar for Public Schools. By Rev. A. C. Clapin, M.A. Fcap. 8vo. 7th Edit. 2*s.* 6*d.*

French Primer. By Rev. A. C. Clapin, M.A. Fcap. 8vo. 3rd Edit. 1*s.*

Primer of French Philology. By Rev. A. C. Clapin. Fcap. 8vo. 1*s.*

Le Nouveau Trésor; or, French Student's Companion. By M. E. S. 16th Edition. Fcap. 8vo. 3*s.* 6*d.*

F. E. A. GASC'S FRENCH COURSE.

First French Book. Fcap 8vo. 76th Thousand. 1*s.* 6*d.*

Second French Book. 37th Thousand. Fcap. 8vo. 2*s.* 6*d.*

Key to First and Second French Books. Fcap. 8vo. 3*s.* 6*d.*

French Fables for Beginners, in Prose, with Index. 14th Thousand. 12mo. 2*s.*

Select Fables of La Fontaine. New Edition. Fcap. 8vo. 3*s.*

Histoires Amusantes et Instructives. With Notes. 13th Thousand. Fcap. 8vo. 2*s.* 6*d.*

Practical Guide to Modern French Conversation. 12th Thousand. Fcap. 8vo. 2s. 6d.

French Poetry for the Young. With Notes. 4th Edition. Fcap. 8vo. 2s.

Materials for French Prose Composition; or, Selections from the best English Prose Writers. 15th Thousand. Fcap. 8vo. 4s. 6d. Key, 6s.

Prosateurs Contemporains. With Notes. 8vo. 6th Edition, revised. 5s.

Le Petit Compagnon; a French Talk-Book for Little Children. 10th Thousand. 16mo. 2s. 6d.

An Improved Modern Pocket Dictionary of the French and English Languages. 30th Thousand, with Additions. 16mo. Cloth. 4s. Also in 2 vols., in neat leatherette, 5s.

Modern French-English and English-French Dictionary. 2nd Edition, revised. In 1 vol. 12s. 6d. (formerly 2 vols. 25s.)

GOMBERT'S FRENCH DRAMA.

Being a Selection of the best Tragedies and Comedies of Molière, Racine, Corneille, and Voltaire. With Arguments and Notes by A. Gombert. New Edition, revised by F. E. A. Gasc. Fcap. 8vo. 1s. each; sewed, 6d.

CONTENTS.

MOLIERE:—Le Misanthrope. L'Avare. Le Bourgeois Gentilhomme. Le Tartuffe. Le Malade Imaginaire. Les Femmes Savantes. Les Fourberies de Scapin. Les Précieuses Ridicules. L'Ecole des Femmes. L'Ecole des Maris. Le Médecin malgré Lui.

RACINE:—Phédre. Esther. Athalie. Iphigénie. Les Plaideurs. La Thébaïde; or, Les Frères Ennemis. Andromaque. Britannicus.

P. CORNEILLE:—Le Cid. Horace. Cinna. Polyeucte.

VOLTAIRE:—Zaïre.

GERMAN CLASS-BOOKS.

Materials for German Prose Composition. By Dr Buchheim. 7th Edition Fcap. 4s. 6d. Key, 3s.

A German Grammar for Public Schools. By the Rev. A. C. Clapin and F. Holl Müller. 2nd Edition. Fcap. 2s. 6d.

Kotzebue's Der Gefangene. With Notes by Dr. W. Stromberg. 1s.

ENGLISH CLASS-BOOKS.

The Elements of the English Language. By E. Adams, Ph.D. 17th Edition. Post 8vo. 4s. 6d.

The Rudiments of English Grammar and Analysis. By E. Adams, Ph.D. New Edition. Fcap. 8vo. 2s.

By C. P. MASON, Fellow of Univ. Coll. London.

First Notions of Grammar for Young Learners. Fcap. 8vo. 8th Thousand. Cloth. 8d.

First Steps in English Grammar for Junior Classes. Demy 18mo. New Edition. 1s.

***Outlines of English* Grammar for the use of Junior Classes.** *7th Edition.* Crown 8vo. 2s.

English Grammar, including the Principles of Grammatical Analysis. 24th Edition. Crown 8vo. 3s. 6d.

A Shorter English Grammar, with copious Exercises. 8th Thousand. Crown 8vo. 3s. 6d.

English Grammar Practice, being the Exercises separately. 1s.

Edited for Middle-Class Examinations.

With Notes on the Analysis and Parsing, and Explanatory Remarks.

Milton's Paradise Lost, Book I. With Life. 3rd Edit. Post 8vo. 2s

——— Book II. With Life. 2nd Edit. Post 8vo. 2s.

——— Book III. With Life. Post 8vo. 2s.

Goldsmith's Deserted Village. With Life. Post 8vo. 1s. 6d.

Cowper's Task, Book II. With Life. Post 8vo. 2s.

Thomson's Spring. With Life. Post 8vo. 2s.

——— Winter. With Life. Post 8vo. 2s.

Practical Hints on Teaching. By Rev. J. Menet, M.A. 5th Edit. Crown 8vo. cloth, 2s. 6d.; paper, 2s.

Test Lessons in Dictation. Paper cover, 1s. 6d.

Questions for Examinations in English Literature. By Rev. W. W. Skeat. 2s. 6d.

Drawing Copies. By P. H. Delamotte. Oblong 8vo. 12s. Sold also in parts at 1s. each.

Poetry for the School-room. New Edition. Fcap. 8vo. 1s. 6d.

Select Parables from Nature, for Use in Schools. By Mrs. A. Gatty. Fcap. 8vo. Cloth. 1s.

School Record for Young Ladies' Schools. 6d.

Geographical Text-Book; a Practical Geography. By M. E. S. 12mo. 2s.
The Blank Maps done up separately, 4to. 2s. coloured.

A First Book of Geography. By Rev. C. A. Johns, B.A., F.L.S. &c. Illustrated. 12mo. 2s. 6d.

Loudon's (Mrs.) Entertaining Naturalist. New Edition. Revised by W. S. Dallas, F.L.S. 5s.

——— **Handbook of Botany.** New Edition, greatly enlarged by D. Wooster. Fcap. 2s. 6d.

The Botanist's Pocket-Book. With a copious Index. By W. R. Hayward. 2nd Edit. revised. Crown 8vo. Cloth limp. 4s. 6d.

Experimental Chemistry, founded on the Work of Dr. Stöckhardt. By C. W. Heaton. Post 8vo. 5s.

Double Entry Elucidated. By B. W. Foster. 12th Edit. 4to.

A New Manual of Book-keeping. By P. Crellin, Accountant. Crown 8vo. 3s. 6d.

Picture School-Books. In Simple Language, with numerous Illustrations. Royal 16mo.

School Primer. 6d.—School Reader. By J. Tilleard. 1s.—Poetry Book for Schools. 1s.—The Life of Joseph. 1s.—The Scripture Parables. By the Rev. J. E. Clarke. 1s.—The Scripture Miracles. By the Rev. J. E. Clarke. 1s.—The New Testament History. By the Rev. J. G. Wood, M.A. 1s.—The Old Testament History. By the Rev. J. G. Wood, M.A. 1s.—The Story of Bunyan's Pilgrim's Progress. 1s.—The Life of Christopher Columbus. By Sarah Crompton. 1s.—The Life of Martin Luther. By Sarah Crompton. 1s.

BOOKS FOR YOUNG READERS.

In 8 vols. Limp cloth, 6*d.* each.

The Cat and the Hen; Sam and his Dog Red-leg; Bob and Tom Lee; A Wreck——The New-born Lamb; Rosewood Box; Poor Fan; Wise Dog——The Three Monkeys——Story of a Cat, told by Herself——The Blind Boy; The Mute Girl; A New Tale of Babes in a Wood——The Dey and the Knight; The New Bank-note; The Royal Visit; A King's Walk on a Winter's Day——Queen Bee and Busy Bee——Gull's Crag, a Story of the Sea.

BELL'S READING-BOOKS.

FOR SCHOOLS AND PAROCHIAL LIBRARIES.

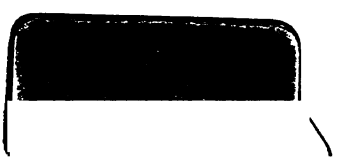
The popularity which the 'Books for Young Readers' have attained is a sufficient proof that teachers and pupils alike approve of the use of interesting stories, with a simple plot in place of the dry combination of letters and syllables, making no impression on the mind, of which elementary reading-books generally consist.

The Publishers have therefore thought it advisable to extend the application of this principle to books adapted for more advanced readers.

Now Ready. Post 8vo. Strongly bound.

Masterman Ready. By Captain Marryat, R.N. 1*s.* 6*d.*
The Settlers in Canada. By Captain Marryat, R.N. 1*s.* 6*d.*
Parables from Nature. (Selected.) By Mrs. Gatty. 1*s.*
Friends in Fur and Feathers. By Gwynfryn. 1*s.*
Robinson Crusoe. 1*s.* 6*d.*
Andersen's Danish Tales. (Selected.) By E. Bell, M.A. 1*s.*
Southey's Life of Nelson. (Abridged.) 1*s.*
Grimm's German Tales. (Selected.) By E. Bell, M.A. 1*s.*
Life of the Duke of Wellington, with Maps and Plans. 1*s.*
Marie; or, Glimpses of Life in France. By A. R. Ellis. 1*s.*
Poetry for Boys. By D. Monro. 1*s.*

Others in Preparation.

BODLEIAN

LONDON:
Printed by STRANGEWAYS & SONS, Tower Street, Upper St. Martin's Lane.